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An Invariant Approach to Symbolic Calculus for Pseudodifferential Operators on Manifolds

Battistotti, Paolo

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AN INVARIANT APPROACH TO
SYMBOLIC CALCULUS FOR
PSEUDODIFFERENTIAL OPERATORS
ON MANIFOLDS

Author:

Paolo Battistotti

Supervisor:

Prof. Yuri Safarov

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KING'S COLLEGE LONDON

Abstract

Faculty of Natural & Mathematical Sciences

Department of Mathematics

Doctor of Philosophy

**An invariant approach to symbolic calculus for pseudodifferential
operators on manifolds**

by Paolo BATTISTOTTI

In this work we present a symbolic calculus for Ψ DOs on a smooth manifold \mathcal{M} based on a suitable notion of the global phase function. In previous literature on the topic, either local coordinates or connections have been used to define the phase functions, symbols $S_{\rho,\delta}^m$ and oscillatory integrals defining Ψ DOs. Traditionally the condition $0 \leq 1 - \rho \leq \delta < \rho \leq 1$ (or at least $0 \leq \delta < \rho \leq 1$ and $\rho > 1/2$) is assumed on the type of the symbol. On the contrary, we rely on a fully analytical notion of the global phase function φ over $(T^*\mathcal{M} \setminus \{0\}) \times \mathcal{M}$. We use φ to define full symbols of Ψ DOs and develop a new version of symbolic calculus on smooth manifolds. All basic results from classical theory remain true under the condition $0 \leq \frac{1-\rho}{2} \leq \delta < \rho \leq 1$ (or at least $0 \leq \delta < \rho \leq 1$ and $\rho > 1/3$). In particular we obtain global formulae for the compositions and adjoints of Ψ DOs. Applications are given to elliptic Ψ DOs, boundedness on Sobolev spaces and functional calculus for elliptic Ψ DOs.

Contents

Abstract	2
Introduction	8
1 Pseudodifferential Operators on Euclidean Spaces	13
1.1 Oscillatory Integrals	16
1.2 The Algebra of Pseudodifferential Operators	21
2 Pseudodifferential Operators on Manifolds	32
2.1 Classical Theory	35
2.2 Modern Literature	44
3 Invariant Approach	51
3.1 Global Phase Functions	53
3.2 φ -symbols	57
3.3 Pseudodifferential Operators	67
4 Stationary Phase Formula	72
4.1 Asymptotic Behaviour	74

4.2	Expansions of Gaussian Integrals	82
4.3	Expansions of Oscillatory Integrals	86
5	φ-symbolic Calculus on Manifolds	90
5.1	φ -equivalence of Amplitudes	92
5.2	Composition Formula	103
6	Applications	114
6.1	Elliptic Operators	116
6.2	Boundedness	118
6.3	Functional Calculus	123
	Bibliography	134

Index of Symbols

α	$= (\alpha_1, \dots, \alpha_n)$ multi-index
$ \alpha $	$= \alpha_1 + \dots + \alpha_n$ length of a multi-index
$\alpha!$	$= \alpha_1! \cdots \alpha_n!$ factorial of a multi-index
$\text{supp } u$	the support of a distribution u
$\text{sing supp } u$	the singular support of a distribution u
χ	(often with indices) various cut-off functions
\mathcal{C}^∞	space of smooth functions
\mathcal{C}_0^∞	space of smooth functions with compact support
$d\theta$	$= (2\pi)^{-n} d\theta$
Δ	Laplace operator
$\langle \theta \rangle$	$= (1 + \theta ^2)^{1/2}$, for $\theta \in \mathbb{R}^n$
\mathbb{N}	$= \{1, 2, \dots\}$ the set of natural numbers
\mathcal{D}	diagonal of a space
\mathcal{C}^k	space of k -differentiable functions

\mathcal{C}_0^k	space of k -differentiable functions with compact support
\mathcal{D}'	Schwartz space of distributions
\mathcal{E}'	space of distributions with compact support
\mathcal{F}	Fourier transform
\mathcal{F}^{-1}	inverse Fourier transform
∇	gradient
∂_x^α	$= \partial_{x_1}^{\alpha_1} \dots \partial_{x_n}^{\alpha_n}$
\mathbb{R}^n	the n -dimensional Euclidean space
\mathcal{S}^m	classical φ -symbols of order m
$\mathcal{S}_{\rho,\delta}^m$	φ -symbols and φ -amplitudes of order m and type (ρ, δ)
$\mathbf{0}$	zero vector
D_x^α	$= (-i)^{ \alpha } \partial_{x_1}^{\alpha_1} \dots \partial_{x_n}^{\alpha_n}$
e_k	unit vector in k -th direction
H^s	Sobolev space $W^{s,2}$
L^2	space of square integrable functions
L^m	classical pseudodifferential operators of order m
$L_{\rho,\delta}^m$	pseudodifferential operators of order m and type (ρ, δ)
S^m	classical symbols of order m
$S_{\rho,\delta}^m$	symbols and amplitudes of order m and type (ρ, δ)

In memory of Yuri Safarov.

(23 January 1958 - 2 June 2015)

Introduction

The theory of pseudodifferential operators (Ψ DOs) was initiated in the '60s and originates from the theory of singular integrals and Fourier analysis. In 1965, Kohn and Nirenberg were the first to give a precise definition of Ψ DOs and its symbolic calculus in [1]. They used Fourier transform to define them. In the same year, Hörmander published another article [2] giving a more intrinsic definition of these operators based on their kernels. In the following years, these operators and their properties were object of intense study. In fact their popularity increased considerably because of their wide range of applications. The importance of this topic is well described by Folland in [3, Chapter 8]:

‘(The theory of Ψ DOs) has become one of the most essential tools in the modern theory of differential equations, as it offers a powerful and flexible way of applying Fourier techniques to the study of variable-coefficient operators and singularities of distributions’.

The third book of the series ‘The analysis of partial differential operators’ by Hörmander ([4–7]) is considered the cornerstone of this theory which contains an extensive and detailed presentation of Ψ DOs in full generality. Other notable books in this field are the works by Trèves ([8, 9]), Taylor ([10]) and Shubin ([11]).

Pseudodifferential operators are completely characterised by a function, traditionally called the symbol. It is a well-known fact that on Euclidean spaces one can construct a canonical linear isomorphism between symbol spaces and corresponding spaces of pseudodifferential operators. Furthermore, one has natural formulae which represent a pseudodifferential operator in terms of its symbol and conversely which give an expression for the symbol of a pseudodifferential operator. By using symbols one gets much insight into the structure of pseudodifferential operators on Euclidean spaces; in particular they give the means to construct the compositions and adjoints of pseudodifferential operators on \mathbb{R}^n . The composition formula is the key to many useful applications, such as boundedness and functional calculus.

Symbolic calculus for pseudodifferential operators on manifolds is not as well-established as the corresponding theory on \mathbb{R}^n . In [12] Pflaum noticed that

‘As the pure consideration of symbols and operators in local coordinates does not reveal the geometry and topology of the manifold one is working on, it is very desirable to build up a general theory of symbols for pseudodifferential operators on manifolds’.

Traditionally, Ψ DOs are defined on manifolds with the use of local coordinates. This approach has some drawbacks. First, only the principal part of the symbol forms a well defined function on the cotangent bundle of the manifold. In other words, modulo lower order terms, a symbol behaves well under change of coordinates. Second, one must impose some conditions on the type of the operator.

Around 1980, Widom was the first to give a proposal for a full symbolic calculus on manifolds in his articles [13, 14]. He put forward the idea to define full symbols of Ψ DOs using geodesics and linear connections. By using a rather general notion of the phase function, Widom constructed a map from the space of pseudodifferential operators on manifolds to the space of symbols and showed, by an abstract argument

for the case of scalar symbols, that this map is bijective modulo smoothing operators and regularising symbols.

In the decade 1996-2005, the method introduced by Widom was developed further. Among others, Pflaum, Safarov and Sharafutdinov re-elaborated the use of differential geometry for defining Ψ DOs and their symbols in their articles [12, 15], [16, 17] and [18, 19] respectively. Safarov managed to obtain some intrinsic and more general results than Pflaum and Sharafutdinov, whose work instead relies on the use of local coordinates.

The aim of this thesis is to develop further the ideas and the methods pushed forward by Safarov. Following an idea proposed by Safarov and McKeag in [17], we propose a fully analytical theory based on an appropriate notion of the phase function for the integral representation of the Ψ DO on the manifold. An important element of innovation with respect to the previous literature is the use of a phase function which is non-linear in the phase variable. Furthermore, it does not depend on the choice of linear connections. We introduce a full symbolic calculus for pseudodifferential operators on manifolds with the following features. First, this new theory is invariant. In fact, even though all calculations are performed in local coordinates, the results do not depend on their choice and are coordinate-free. Second, the composition formula has a concrete representation in form of an asymptotic expansion.

Similar to Safarov's approach, the definition of the symbol is intertwined with the concept of the phase function. However, this definition of symbols coincides and extends the classical one. We adopt the same outline as [16] and present a theory for the symbolic calculus of Ψ DOs of Hörmander type on manifolds with many elements of innovation. We also discuss some applications deriving from such a theory. A detailed structure of the work follows.

In the first chapter, we briefly recall the theory of Ψ DO on Euclidean spaces. We emphasise particularly the symbolic calculus of Ψ DOs on \mathbb{R}^n . We discuss the classical results concerning the compositions and the adjoints of Ψ DOs, which we aim to extend from the setting of Euclidean spaces to the one of manifolds.

In Chapter 2, we present the well-established results concerning the symbolic calculus on manifolds. Particularly, we recall the theory based on the use of local coordinates. In addition, we review the literature investigating this subject originating from the work by Widom introduced briefly above.

In the third chapter, we introduce the intrinsic integral representation for a Ψ DO on manifolds. We rely on this representation based on the use of an appropriate notion of the global phase function in order to develop a new version of symbolic calculus for Ψ DOs. Safarov and McKeag put forward this idea in [17, Section 7.8]. New classes of symbols, whose definitions are based on the notion of the global phase function, are also introduced. These are examined in detail. We prove properties concerning their structure consistent with previous results.

Chapter 4 revisits the stationary phase formula. We prove a version of this well-established method adjusted for oscillatory integrals. We aim to apply it to the classes of pseudodifferential operators introduced previously in order to derive the adjoint and composition formulae. The results contained in this chapter are crucial for developing symbolic calculus in our setting. We base our analysis on the proof presented in [4, Section 7.7] since it is more effective for computing the terms of the expansion.

The fifth chapter contains the core of the work. The central result of this thesis is the achievement of the formula for the full symbol of compositions and adjoints of pseudodifferential operators. The asymptotic expansions describing the symbols

maintain the elegance and simplicity of the well-known version for Euclidean spaces. We emphasise that this new approach allows one to consider cases otherwise excluded in the classical approach on manifolds.

Finally, the symbolic calculus introduced gives us the means to build up applications concerning elliptic operators, boundedness and functional calculus. We generalise the classical results on parametrices for elliptic Ψ DOs and boundedness on Sobolev spaces. We also extend the results concerning functional calculus contained in Section 11 of [16], valid only for the Laplace operator, to the broader class of elliptic classical pseudodifferential operators.

Chapter 1

Pseudodifferential Operators on Euclidean Spaces

In this chapter, we recall the classical theory of pseudodifferential operators on Euclidean spaces and set the notation. We will refer to these basic definitions and results presented here later on. The discussion of the topics is schematic since this theory is well-established. A more exhaustive and extended discussion of the topics presented in this section may be found in any of the textbooks mentioned in the introduction.

First, we present oscillatory integrals. Because of their abstract nature, they have only direct applications. In fact, they lay the foundation for the theory of pseudodifferential operators. They are not integrals in the common sense, but rather in a distributional sense. Therefore, their precise definition and regularisation require much attention.

In the second section, we formally introduce pseudodifferential operators and their basic properties. We draw our attention particularly to symbolic calculus for pseudodifferential operators.

As already mentioned in the introduction, each Ψ DO can be identified with an essentially unique smooth function called the symbol. The uniqueness of the symbol comes at a price. In fact, rather than having a compact representation for the symbol we need to express it as a series of terms consisting of decreasing order. Obviously the most interesting element in this sequence is the first one because it is the one with highest order. Moreover, when we go further in the series, the terms contain less and less information since they start decreasing polynomially to zero at infinity faster and faster. As a result, the correspondence between the set of symbols and the set of pseudodifferential operators is bijective modulo very regular factors, or equivalently we have a bijection only between factor classes of the two sets.

The asymptotic expansion of a symbol is obtained from a less general function also describing the Ψ DO called the amplitude. Although we are usually able to express an amplitude by means of just one function, rather than a series, the class of such elements has the disadvantage of not being in one to one correspondence with the class of Ψ DOs.

Symbolic calculus is the study of the correspondence between the structures of the class of symbols and the class of Ψ DOs. One can prove easily that these two sets are $*$ -algebras. In the set of symbols it is enough to consider as multiplication and involution pointwise multiplication and complex conjugate respectively; instead, in the set of Ψ DOs we ought to consider operator composition and conjugation respectively. The two classes are isomorphic as vector spaces. However, the multiplicative and involutive structure are not preserved via this correspondence.

After all, the algebra of symbols is commutative while the algebra of operators is not.

In general, the symbols of the compositions and adjoints of Ψ DOs are not the products or the adjoints respectively of the symbols of the original Ψ DOs. Nonetheless one can find expressions for these symbols by some simple manipulations based on Taylor's formula. One can prove that the two classes are isomorphic as $*$ -algebras if we neglect all but the highest order terms in their asymptotic expansions. However, this is not enough to define the full symbol of the composition. In some cases, we must keep track of the remaining infinite terms in the series. Although they are less interesting in terms of properties because of the lower order, they are necessary to fully describe the symbol of a Ψ DO.

These results concerning the composition and adjoint of Ψ DOs and their symbols have a central role in the theory of pseudodifferential operators. In fact, they are the key for many fundamental applications such as boundedness and functional calculus. They allow us to tackle the problem dealing with symbols, which are smooth functions, rather than with pseudodifferential operators, which are oscillatory integrals.

Let us assume once and for all that $m, \rho, \delta \in \mathbb{R}$ are real numbers such that $0 \leq \delta < \rho \leq 1$. We denote by $\delta' = \max\{1 - \rho, \delta\}$. A multi-index α is an ordered n -tuple of nonnegative integers $\alpha = (\alpha_1, \dots, \alpha_n)$. We use $|\alpha|$ and $\alpha!$ to denote the size $\alpha_1 + \dots + \alpha_n$ and the product of the factorials of its entries $\alpha_1! \dots \alpha_n!$ respectively. If $f \in \mathcal{C}^{|\alpha|}$, then $\partial_x^\alpha f(x)$ denotes the derivative $\partial_{x_1}^{\alpha_1} \dots \partial_{x_n}^{\alpha_n} f(x)$. Finally, for $\theta \in \mathbb{R}^n$, the notation $\langle \theta \rangle$ indicates the factor $(1 + |\theta|^2)^{1/2}$ and $d\theta = (2\pi)^{-n} d\theta$.

1.1 Oscillatory Integrals

We start by introducing the concepts of amplitude and phase function. They are the bricks required to define formally an oscillatory integral, which is not an integral in the usual sense but rather in a distributional sense. We clarify the meaning of this at a later stage. Oscillatory integrals generate an extremely general class of objects. However, by adding some additional conditions on the properties of the amplitudes or of the phase functions, we obtain operators of fundamental importance, such as pseudodifferential or Fourier integral operators.

Definition 1.1. Let $N \in \mathbb{N}$ and let W be an open set in \mathbb{R}^N . An amplitude is a function $a \in \mathcal{C}^\infty(W \times \mathbb{R}^n)$ such that for all multi-indices α and β and for all compact sets $K \subset W$ there exists $C = C_{\alpha,\beta,K}$ which satisfies

$$|\partial_\theta^\alpha \partial_w^\beta a(w, \theta)| \leq C \langle \theta \rangle^{m-\rho|\alpha|+\delta|\beta|} \quad \text{for } (w, \theta) \in K \times \mathbb{R}^n. \quad (1.1)$$

The set of amplitudes will be denoted by $S_{\rho,\delta}^m(W \times \mathbb{R}^n)$, m is called the order and (ρ, δ) the type of the amplitudes.

One can observe that $S_{\rho_1,\delta_1}^{m_1}(W \times \mathbb{R}^n) \subseteq S_{\rho_2,\delta_2}^{m_2}(W \times \mathbb{R}^n)$ if $\rho_1 \geq \rho_2$ and $\delta_1 \leq \delta_2$ and $m_1 \leq m_2$. Furthermore, for all multi-indices α and β , we have $\partial_\theta^\alpha \partial_w^\beta a \in S_{\rho,\delta}^{m-\rho|\alpha|+\delta|\beta|}(W \times \mathbb{R}^n)$ when $a \in S_{\rho,\delta}^m(W \times \mathbb{R}^n)$.

Define the set of regularising amplitudes by

$$S^{-\infty}(W \times \mathbb{R}^n) = \bigcap_{m \in \mathbb{R}} S_{1,0}^m(W \times \mathbb{R}^n).$$

The set of smooth functions with compact support and the set of Schwartz functions constitute the core of the set of regularising amplitudes, i.e. $\mathcal{C}_0^\infty(\mathbb{R}^n) \subseteq \mathcal{S}(\mathbb{R}^n) \subseteq S^{-\infty}(W \times \mathbb{R}^n)$.

The parameters m , ρ and δ play a central role in describing an amplitude. In particular, they give us information about the behaviour of the amplitude when $|\theta|$ is large and the rate of change of it induced by differentiation.

The space $S_{\rho,\delta}^m(W \times \mathbb{R}^n)$ is endowed with a natural convex topology. Let K be a compact set in W , multi-indices α, β and $a \in S_{\rho,\delta}^m(W \times \mathbb{R}^n)$, we define

$$p_{K,\alpha,\beta}(a) = \sup_{(w,\theta) \in K \times \mathbb{R}^n} |\partial_\theta^\alpha \partial_w^\beta a(w, \theta)| \langle \theta \rangle^{-m+\rho|\alpha|-\delta|\beta|}.$$

It is seen at once that $p_{K,\alpha,\beta,\gamma}$ is a seminorm on $S_{\rho,\delta}^m(W \times \mathbb{R}^n)$ and defines the topology of this space when K ranges over the collection of all compact subsets of W and all multi-indices α, β .

It can also be proved that the space $S_{\rho,\delta}^m(W \times \mathbb{R}^n)$ is complete with respect to these seminorms, therefore $S_{\rho,\delta}^m(W \times \mathbb{R}^n)$ is a Fréchet space.

Now, we present two examples of amplitudes. Homogeneous functions constitute the most notable set of amplitudes. The second example shows that there exist cases where the type of the amplitude is not $(1, 0)$.

Example 1.1. Let $h(w, \theta) \in \mathcal{C}^\infty(W \times \mathbb{R}^n)$ be a positively homogeneous function of degree m for large values in θ (i.e. $\exists C > 0$ such that $h(w, \lambda\theta) = \lambda^m h(w, \theta)$ for all $|\theta| > C$ and $\lambda > 0$), then $h \in S_{1,0}^m(W \times \mathbb{R}^n)$.

Example 1.2. Let $a(w, \theta) = \sin(\langle w \rangle \langle \theta \rangle^\rho)$, then $a \in S_{1-\rho, \rho}^0(W \times \mathbb{R}^n)$. In fact, to prove inequality (1.1), we only need to observe

$$\partial_{\theta_j} \langle \theta \rangle^\rho = \rho \theta_j \langle \theta \rangle^{\rho-2} \sim C \langle \theta \rangle^{\rho-1} \quad \text{as } |\theta| \rightarrow \infty.$$

Definition 1.2. We say that a real valued function $\varphi \in \mathcal{C}^\infty(W \times \mathbb{R}^n \setminus \{0\})$ is a phase function if φ is positively homogeneous of degree 1 in θ and $\nabla_{w, \theta} \varphi(w, \theta) \neq 0$ for all $(w, \theta) \in W \times (\mathbb{R}^n \setminus \{0\})$.

Now, we have the tools required to introduce formally the definition of an oscillatory integral.

Definition 1.3. An integral of the form

$$\int_{\mathbb{R}^n} \int_W e^{i\varphi(w, \theta)} a(w, \theta) u(w) dw d\theta, \quad u \in \mathcal{C}_0^\infty(W), \quad (1.2)$$

where $a \in S_{\rho, \delta}^m(W \times \mathbb{R}^n)$ and φ is a phase function, is called an oscillatory integral.

Notwithstanding the name, this is not an integral in the common sense. In general, integral (1.2) it is not absolutely convergent. Instead, it is an integral in a distributional sense. Now, we clarify the meaning of this non-standard object. In particular, we present a method to regularise it based on integration by parts. What follows until the end of the section is an adaptation of some parts of [11, Section 1].

Lemma 1.4. Given a phase function φ , there exists an operator

$$L = \sum_{j=1}^n a_j(w, \theta) \partial_{\theta_j} + \sum_{k=1}^N b_k(w, \theta) \partial_{w_k} + c(w, \theta),$$

with $a_j \in S_{1,0}^0(W \times \mathbb{R}^n)$, $b_k \in S_{1,0}^{-1}(W \times \mathbb{R}^n)$ and $c \in S_{1,0}^{-1}(W \times \mathbb{R}^n)$, such that:

a) $Le^{i\varphi} = e^{i\varphi}$;

b) There exists $\tilde{a}_j \in S_{1,0}^0(W \times \mathbb{R}^n)$, $\tilde{b}_k \in S_{1,0}^{-1}(W \times \mathbb{R}^n)$ and $\tilde{c} \in S_{1,0}^{-1}(W \times \mathbb{R}^n)$, such that

$$L^t = \sum_{j=1}^n \tilde{a}_j(w, \theta) \partial_{\theta_j} + \sum_{k=1}^N \tilde{b}_k(w, \theta) \partial_{w_k} + \tilde{c}(w, \theta);$$

c) $(L^t)^t = L$.

Proof. Let

$$\Xi(w, \theta) = \left(\sum_{j=1}^n |\theta|^2 |\partial_{\theta_j} \varphi|^2 + \sum_{k=1}^N |\partial_{w_k} \varphi|^2 \right)$$

Notice that $\Xi \in \mathcal{C}^\infty(W \times (\mathbb{R}^n \setminus \{0\}))$ is positively homogeneous of degree 2 in θ . In order to get rid of the singularity at 0 we introduce the cut off function $\chi \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ such that

$$\chi(\theta) = \begin{cases} 1 & |\theta| \leq 1/4 \\ 0 & |\theta| \geq 1/2. \end{cases}$$

Define

$$L = \sum_{j=1}^n a_j(w, \theta) \partial_{\theta_j} + \sum_{k=1}^N b_k(w, \theta) \partial_{w_k} + c(w, \theta),$$

where

$$a_j(w, \theta) = -i[1 - \chi(\theta)] \frac{|\theta|^2 \partial_{\theta_j} \varphi(w, \theta)}{\Xi(w, \theta)};$$

$$b_k(w, \theta) = -i[1 - \chi(\theta)] \frac{\partial_{w_k} \varphi(w, \theta)}{\Xi(w, \theta)};$$

$$c(w, \theta) = \chi(\theta).$$

Since Ξ is positively homogeneous of degree 2 in θ , φ is positively homogeneous of degree 1 in θ and $1 - \chi$ is positively homogeneous of degree 0 in θ , the coefficients

$a_j \in S_{1,0}^0(W \times \mathbb{R}^n)$, $b_k \in S_{1,0}^{-1}(W \times \mathbb{R}^n)$ and $c \in S_{1,0}^{-1}(W \times \mathbb{R}^n)$ as required. Properties a), b) and c) are the consequence of elementary calculations.

For example, we prove a). Since

$$\partial_{\theta_j} e^{i\varphi} = i e^{i\varphi} \partial_{\theta_j} \varphi, \quad \partial_{w_k} e^{i\varphi} = i e^{i\varphi} \partial_{w_k} \varphi,$$

it is immediate that

$$\left(\sum_{j=1}^n a_j(w, \theta) \partial_{\theta_j} + \sum_{k=1}^N b_k(w, \theta) \partial_{w_k} \right) e^{i\varphi} = [1 - \chi(\theta)] e^{i\varphi}$$

and therefore conclude that $Le^{i\varphi} = e^{i\varphi}$. □

We are now ready to discuss the regularisation of

$$\int_{\mathbb{R}^n} \int_W e^{i\varphi(w, \theta)} a(w, \theta) u(w) dw d\theta, \quad u \in \mathcal{C}_0^\infty(W), \quad (1.3)$$

where $a \in S_{\rho, \delta}^m(W \times \mathbb{R}^n)$ and φ is a phase function. First we observe that

$$|e^{i\varphi(w, \theta)} a(w, \theta) u(w)| \leq C \langle \theta \rangle^m,$$

uniformly for $w \in \text{supp } u$. Therefore, the integral in (1.3) converges absolutely if $m < -n$. In the case this condition does not hold, we can still reduce the integral to the situation above via integration by parts. By Lemma 1.4, we can substitute $e^{i\varphi}$ with $L^k e^{i\varphi}$, $k \in \mathbb{N}$, in (1.2). Hence, the integral in (1.3) becomes

$$\begin{aligned} \int_{\mathbb{R}^n} \int_W e^{i\varphi(w, \theta)} a(w, \theta) u(w) dw d\theta &= \int_{\mathbb{R}^n} \int_W L^k e^{i\varphi(w, \theta)} a(w, \theta) u(w) dw d\theta \\ &= \int_{\mathbb{R}^n} \int_W e^{i\varphi(w, \theta)} (L^t)^k [a(w, \theta) u(w)] dw d\theta. \end{aligned}$$

Since $L^k[a(w, \theta)u(w)] \in S_{1,0}^{m-k\delta'}(W \times \mathbb{R}^n)$, we only need to chose k large enough so that $m - k\delta' < -n$. This makes the integral in the right-hand side absolutely convergent.

This regularisation method based on integration by parts is commonly used in the theory of pseudodifferential operators for multiple purposes. We will make use of a similar technique at a later stage.

We conclude this section by pointing out that the domain of a Ψ DO can be easily extended from $\mathcal{C}_0^\infty(W)$ to the set of distributions with compact support $\mathcal{E}'(W)$ by continuity and duality.

1.2 The Algebra of Pseudodifferential Operators

As mentioned in the previous section, the class of oscillatory integrals is far too broad to give rise to an interesting theory. Therefore, only subsets of this class are considered. Additional requirements on the structure of the integrals or properties of the phase function and amplitudes generates sets of operators commonly used in mathematics.

An example is the class of Fourier integral operators. This theory considers operators $A : \mathcal{E}'(Y) \rightarrow \mathcal{D}'(X)$ that can be represented in the form

$$Au(x) = \int_{\mathbb{R}^n} \int_Y e^{i\varphi(x,y,\theta)} a(x,y,\theta) u(y) dy d\theta, \quad (1.4)$$

where X and Y are open sets in \mathbb{R}^n , φ is a phase function and a is an amplitude. Operators represented in this form allows us to study the propagation of singularities of the distribution u . The phase function φ , or more precisely its gradient with

respect to θ , tells us where the singularities propagate and the amplitude a , or more precisely its order and the set where this function vanishes, tells us how they propagate. Precise details on Fourier integral operators can be found for example in [9].

We are interested in studying pseudodifferential operators. These are a special case of Fourier integral operators. They have the additional requirement that the phase function coincides with $\varphi(x, y, \theta) = \langle x - y, \theta \rangle$. Note that the function $\langle x - y, \theta \rangle$ is a phase function because all the conditions of Definition 1.2 are satisfied. The smaller generality of this integral allows us to obtain stronger results in terms of regularity of the operator and the manipulation of the amplitude. In this section, we briefly recall these results. For an extended discussion of the topic one can refer to [6] or [8].

First, we define the class of amplitudes. The following definition is a repetition of Definition 1.1 adjusted to the new setting.

Definition 1.5. Let X be an open set in \mathbb{R}^n . An amplitude is a function $a \in \mathcal{C}^\infty(X \times X \times \mathbb{R}^n)$ such that for all multi-indices α, β and γ and for all compact sets $K \subset X \times X$ there exists $C = C_{\alpha, \beta, \gamma, K}$ which satisfies

$$|\partial_\theta^\alpha \partial_x^\beta \partial_y^\gamma a(x, y, \theta)| \leq C \langle \theta \rangle^{m - \rho|\alpha| + \delta(|\beta| + |\gamma|)} \quad \text{for } (x, y, \theta) \in K \times \mathbb{R}^n.$$

We denote by $S_{\rho, \delta}^m(X \times X \times \mathbb{R}^n)$ the set of amplitude defined on the set $X \times X \times \mathbb{R}^n$, m is called the order and (ρ, δ) the type of the amplitude.

Obviously, the amplitudes in this setting have the same properties as the ones previously presented in Section 1.1. Here, we just repeat them for the sake of clarity.

We observe that $S_{\rho_1, \delta_1}^{m_1}(X \times X \times \mathbb{R}^n) \subseteq S_{\rho_2, \delta_2}^{m_2}(X \times X \times \mathbb{R}^n)$ if $\rho_1 \geq \rho_2$ and $\delta_1 \leq \delta_2$ and $m_1 \leq m_2$. Furthermore, for all multi-indices α, β and γ we have $\partial_\theta^\alpha \partial_x^\beta \partial_y^\gamma a \in S_{\rho, \delta}^{m - \rho|\alpha| + \delta(|\beta| + |\gamma|)}(X \times X \times \mathbb{R}^n)$ for any $a \in S_{\rho, \delta}^m(X \times X \times \mathbb{R}^n)$.

Define the class of regularising amplitudes by

$$S^{-\infty}(X \times X \times \mathbb{R}^n) = \bigcap_{m \in \mathbb{R}} S_{1,0}^m(X \times X \times \mathbb{R}^n).$$

Definition 1.6. An integral of the form

$$Au(x) = \int_{\mathbb{R}^n} \int_X e^{i\langle x-y, \theta \rangle} a(x, y, \theta) u(y) dy d\theta, \quad u \in \mathcal{C}_0^\infty(X), \quad (1.5)$$

where $a \in S_{\rho, \delta}^m(X \times X \times \mathbb{R}^n)$, is called a pseudodifferential operator of order m and type (ρ, δ) . We denote by $L_{\rho, \delta}^m(X)$ the class of such operators. In particular, $L^{-\infty}(X)$ denotes the class of such operators whose amplitude is in $S^{-\infty}(X \times X \times \mathbb{R}^n)$.

Note that (1.5) is an oscillatory integral. In a similar fashion to the case of oscillatory integrals, integral (1.5) is not in general absolutely convergent. In order to make sense of it, we need to regularise it. The method, up to some minor adjustments, is the same used in Section 1.1 for oscillatory integrals (see Lemma 1.4).

Remark 1.7. Fourier integral operators and pseudodifferential operators are powerful tools for investigating the singularities of distributions and the regularity of solutions of partial differential operators. Because of this, we are only interested in considering operators that maintain or provide interesting information regarding the singularities of a distribution. It is therefore meaningful to disregard operators smoothening all singularities. An operator that cancels all the singularities turning a distribution into a smooth function is of no interest for this theory. Therefore,

the theory is developed modulo the oscillatory integrals generating operators with such a property, usually called smoothing operators.

Definition 1.8. An operator $T : \mathcal{E}'(X) \mapsto \mathcal{C}^\infty(X)$ is called a smoothing operator.

A direct consequence of this fact is that, considering a pseudodifferential operator represented in the form (1.7), we can always assume that the amplitude $a(x, y, \theta)$ vanishes in a neighbourhood of $\theta = 0$.

In fact, if $\chi \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ such that

$$\chi(\theta) = \begin{cases} 1 & |\theta| < 1/4 \\ 0 & |\theta| \geq 1/2, \end{cases}$$

then

$$\begin{aligned} & \int \int e^{i(x-y) \cdot \theta} a(x, y, \theta) u(y) dy d\theta \\ &= \int \int e^{i(x-y, \theta)} \chi(\theta) a(x, y, \theta) u(y) dy d\theta \\ & \quad + \int \int e^{i(x-y, \theta)} (1 - \chi(\theta)) a(x, y, \theta) u(y) dy d\theta. \end{aligned} \tag{1.6}$$

In order to conclude, we just need to notice that, since $\chi(\theta)a(x, y, \theta)$ is compactly supported in θ , the first term of the right side of equality (1.6) generates a smooth function (negligible), and

$$(1 - \chi(\theta))a(x, y, \theta) = 0, \quad \text{when } |\theta| < 1/4.$$

This technique, already used in the proof of Lemma 1.4, will be from now on implicitly used.

This means for example that any positively homogeneous function $a(x, y, \theta) \in$

$\mathcal{C}^\infty(X \times X \times (\mathbb{R}^n \setminus \{0\}))$ of degree m in θ belongs to $S_{1,0}^m(X \times X \times \mathbb{R}^n)$ (see also Example 1.1). In particular a phase function φ is in $S_{1,0}^1(X \times X \times \mathbb{R}^n)$.

The following proposition gives an account of all the most important features of a Ψ DO.

Proposition 1.9. *Let A be a Ψ DO given by the formula*

$$Au(x) = \int_{\mathbb{R}^n} \int_X e^{i\langle x-y, \theta \rangle} a(x, y, \theta) u(y) dy d\theta, \quad (1.7)$$

with $a \in S_{\rho, \delta}^m(X \times X \times \mathbb{R}^n)$. Let \mathcal{A} be the kernel of A and \mathcal{D} the diagonal in $X \times X$. Then

- a) $\mathcal{A} \in \mathcal{C}^\infty((X \times X) \setminus \mathcal{D})$;
- b) A is pseudolocal, i.e. $\text{sing supp } Au \subseteq \text{sing supp } u$;
- c) If $a \in S^{-\infty}(X \times X \times \mathbb{R}^n)$, then

$$\int_{\mathbb{R}^n} e^{i\langle x-y, \theta \rangle} a(x, y, \theta) d\theta \in \mathcal{C}^\infty(X \times X).$$

- d) If $a \in S(X \times X \times \mathbb{R}^n)$ vanishes for $x = y$, then there exists $b \in S_{\rho, \delta}^{m-\rho+\delta}(X \times X \times \mathbb{R}^n)$ such that

$$\int_{\mathbb{R}^n} \int_X e^{i\langle x-y, \theta \rangle} a(x, y, \theta) u(y) dy d\theta = \int_{\mathbb{R}^n} \int_X e^{i\langle x-y, \theta \rangle} b(x, y, \theta) u(y) dy d\theta.$$

- e) A defines continuous linear maps

$$A : \mathcal{C}_0^\infty(X) \rightarrow \mathcal{C}^\infty(X),$$

$$A : \mathcal{C}'(X) \rightarrow \mathcal{D}'(X).$$

We now analyse individually the properties of Proposition 1.9.

Property a) is consistent with what we mentioned in Section 1.1 regarding the propagation of singularities and the role of the phase function. In fact, all the singularities propagate on the diagonal $\mathcal{D} = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n : x = y\}$, which is the space obtained by equating to zero the gradient with respect to θ of the phase function $\langle x - y, \theta \rangle$.

Property b) tells us that the set of singularities of a distribution is not increased by the action of a pseudodifferential operator. The pseudolocality of Ψ DOs is crucial for applications.

Property c) highlights the fact that any operator in $L^{-\infty}(X)$ is a smoothing operator. It is interesting to note that not every smoothing operator is a Ψ DO, however. For example if $a(x, y, \theta) = \chi(x)\psi(\theta)$ where $\chi \in \mathcal{C}^\infty(\mathbb{R}^n)$ and $\psi \in \mathcal{E}'(\mathbb{R}^n)$ then the operator A defined by (1.7) is smoothing. In fact, $Au = \chi(u * \mathcal{F}^{-1}\psi)$ which is in \mathcal{C}^∞ since $\mathcal{F}^{-1}\psi$ is in \mathcal{C}^∞ , but a does not belong to our symbol classes unless ψ is \mathcal{C}^∞ .

Property d), in addition to reinforcing what was stated in a), reveals an interesting fact about the correspondence between amplitudes and Ψ DOs: two different amplitudes not necessarily of the same order can define the same Ψ DO. Furthermore, we can easily notice that for any given pseudodifferential operator there correspond infinitely many different amplitudes. For example, let $\chi_1, \chi_2 \in \mathcal{C}^\infty(X)$ such that $\text{supp}(\chi_1) \cap \text{supp}(\chi_2) = \emptyset$, then

$$\int_{\mathbb{R}^n} \int_X e^{i\langle x-y, \theta \rangle} \chi_1(x) \chi_2(y) u(y) dy d\theta = 0. \quad (1.8)$$

Finally, Looking at property e) in Proposition 1.9, we notice that one drawback of pseudodifferential operators is that their domains consist of compactly supported

functions or distributions while their ranges do not, so in general they cannot be composed with one another. It is therefore often convenient to impose an additional condition on them to overcome this difficulty.

Definition 1.10. A Ψ DO A with Schwartz kernel \mathcal{A} is said to be properly supported if both projections

$$\begin{array}{ccc} \pi_1 : \text{supp } \mathcal{A} & \rightarrow & X \\ (x, y) & \rightarrow & x \end{array} \quad \text{and} \quad \begin{array}{ccc} \pi_2 : \text{supp } \mathcal{A} & \rightarrow & X \\ (x, y) & \rightarrow & y \end{array}$$

are proper maps. Recall that a continuous map is called proper if the inverse image of any compact set is compact.

Proposition 1.11. *Let A be a properly supported Ψ DO. Then A defines a continuous linear map*

$$A : \mathcal{C}_0^\infty(X) \rightarrow \mathcal{C}_0^\infty(X),$$

which extends by continuity to a continuous linear map

$$A : \mathcal{E}'(X) \rightarrow \mathcal{E}'(X).$$

Here continuity is understood to be in the sense of weak topologies on $\mathcal{E}'(X)$.

Remark 1.12. Given a Ψ DO, it is always possible to find another one with the same order and type that is properly supported such that the difference of the two is a smoothing operator.

We are now in a good position to start discussing the symbolic calculus of Ψ DOs in \mathbb{R}^n . Much of the calculus of Ψ DOs consists of performing calculations with ‘less regular terms’ (higher order) and keeping careful track of ‘more regular terms’

(lower order). For this reason, the following notion of asymptotic expansion is crucial.

Definition 1.13. Let $a_j \in S_{\rho,\delta}^{m_j}(X \times X \times \mathbb{R}^n)$, where $m_j \in \mathbb{R}$, $m_j \rightarrow -\infty$ and $a \in S_{\rho,\delta}^{m_1}(X \times \mathbb{R}^n)$. We will write

$$a(x, y, \theta) \sim \sum_{j=0}^{\infty} a_j(x, y, \theta)$$

and we will say that $\sum_{j=0}^{\infty} a_j$ is an asymptotic expansion of a , if for any integer $N \geq 1$ we have

$$a(x, y, \theta) - \sum_{j=1}^N a_j(x, y, \theta) \in S_{\rho,\delta}^{l_N}(X \times \mathbb{R}^n),$$

with $l_N \rightarrow -\infty$.

We emphasise that the series $\sum_{j=1}^{\infty} a_j$ does not have to be convergent, and if it does converge the limit does not have to be a . The sum $\sum_{j=1}^{\infty} a_j(x, y, \theta)$ is simply a convenient way of keeping track of the sequence $\{a_j\}_{j=1}^{\infty}$.

In general, all equalities concerning amplitudes are assumed to be modulo $S^{-\infty}(X \times X \times \mathbb{R}^n)$. This does not affect the action of the operator on the singularities since such amplitudes give rise to smoothing operators, which are negligible.

It is important to highlight the following result concerning asymptotic expansions.

Lemma 1.14. Let $a_j \in S_{\rho,\delta}^{m_j}(X \times X \times \mathbb{R}^n)$ where $j \in \mathbb{N}$, $m_{j+1} < m_j$, $m_j \rightarrow -\infty$. Then there exists a function $a \in S_{\rho,\delta}^{m_1}(\varphi, \mathcal{M})$ unique modulo $S^{-\infty}(X \times \mathbb{R}^n)$, such that $a \sim \sum_k a_k$.

This lemma shows that an asymptotic expansion can be used to build amplitudes.

The proposition that we are about to state shows that it is always possible to find an asymptotic expansion independent of the variable y defining a given amplitude a .

In other words, applying Lemma 1.14, we can build a new amplitude, which depends only on x and θ , defining the same operator of a using (1.7). The amplitudes depending only on variables x and θ are traditionally called symbols. They are of fundamental importance because they are essentially in bijective correspondence with the class of Ψ DOs.

Proposition 1.15. *Let A be a properly supported Ψ DO whose representation is as in (1.7), then*

$$a(x, y, \theta) \sim \sum_{\alpha} \frac{(-i)^{|\alpha|}}{\alpha!} \partial_{\theta}^{\alpha} \partial_y^{\alpha} a(x, y, \theta) \Big|_{y=x}. \quad (1.9)$$

Remark 1.16. Notice that

$$D_{\theta}^{\alpha} \partial_y^{\alpha} a(x, y, \theta) \Big|_{y=x} \in S_{\rho, \delta}^{m-|\alpha|(\rho-\delta)}(X \times \mathbb{R}^n).$$

Therefore, this is a well defined asymptotic expansion, since we assume $\rho > \delta$.

It is also sometimes convenient to consider smaller classes of Ψ DOs. Therefore, we also introduce the notions of classical amplitudes and, by analogy, of classical symbols.

Definition 1.17. We denote by $S^m(X \times X \times \mathbb{R}^n)$ the class of amplitudes $a \in S_{1,0}^m$ that admits an asymptotic expansion

$$a(x, y, \theta) \sim \sum_{j=0}^{\infty} a_{m-j}(x, y, \theta),$$

with a_{m-j} positively homogeneous in θ of degree $m - j$. If a is independent of y , it is called a classical symbol and for simplicity we will denote this class again by S^m . In this case the leading homogeneous term a_m is called the principal symbol.

An important example of classical symbols is the symbols of partial differential equations with constant coefficients. The symbols of such operators are polynomials in θ with constant coefficients, e.g. the partial differential operator $P(\partial) = \sum_{|\alpha| \leq m} c_\alpha \partial^\alpha$, $c_\alpha \in \mathbb{C}$, can be written in the integral representation (1.7) having $P(\theta) = \sum_{|\alpha| \leq m} (-i)^{|\alpha|} c_\alpha \theta^\alpha$ as its symbol. In this case, $P_m(\theta) = \sum_{|\alpha|=m} (-i)^{|\alpha|} c_\alpha \theta^\alpha$ is the principal symbol.

We are now ready to present the formula for the symbol of the composition of two pseudodifferential operators and of the adjoint of a pseudodifferential operator. These constitute the core results for the symbolic calculus theory on Euclidean spaces.

Proposition 1.18. *Let A and B be two properly supported Ψ DOs and let their symbols be $\sigma_A(x, \theta) \in S_{\rho, \delta}^{m_1}(X \times \mathbb{R}^n)$ and $\sigma_B(x, \theta) \in S_{\rho, \delta}^{m_2}(X \times \mathbb{R}^n)$ respectively. The composition AB is a properly supported Ψ DO, whose symbol is in $S_{\rho, \delta}^{m_1+m_2}(X \times \mathbb{R}^n)$ and satisfies the relation*

$$\sigma_{AB}(x, \theta) \sim \sum_{\alpha} \frac{(-i)^{|\alpha|}}{\alpha!} \partial_\theta^\alpha \sigma_A(x, \theta) \partial_x^\alpha \sigma_B(x, \theta). \quad (1.10)$$

Proposition 1.19. *Let A be a properly supported Ψ DO and let his symbol be $\sigma_A(x, \theta) \in S_{\rho, \delta}^m(X \times \mathbb{R}^n)$ a. The adjoint A^* is a properly supported Ψ DO, whose symbol is in $S_{\rho, \delta}^m(X \times \mathbb{R}^n)$ and satisfies the relation*

$$\sigma_{A^*}(x, \theta) \sim \sum_{\alpha} \frac{(-i)^{|\alpha|}}{\alpha!} \partial_\theta^\alpha \partial_x^\alpha \overline{\sigma_A(x, \theta)}. \quad (1.11)$$

Obviously, formulae (1.10) and (1.11) constitute key results for the symbolic calculus of pseudodifferential operators.

These prove that the compositions and adjoints of Ψ DOs are again Ψ DOs. The order of the composition is the sum of the orders of the two operators considered, while the adjoint of a Ψ DO has the same order as the original operator. Therefore, the composition does not increase the order. Adjoints and compositions behave as one expects.

We mentioned above that the class of amplitudes and pseudodifferential operators are $*$ -algebras. They are isomorphic if we regard them just as vector spaces, neglecting the multiplicative and involutive structures. However, the two sets become isomorphic as $*$ -algebras if we neglect all but the highest order terms in the asymptotic expansion (1.10).

Finally, formulae (1.10) and (1.11) are the fundamental tools for the proofs of all basic applications on Ψ DOs. For example, classical results on the boundedness of these operators rely strongly on it. Even the theory of functional calculus of partial differential operators makes heavy use of the properties deriving from Propositions 1.18 and 1.19.

Chapter 2

Pseudodifferential Operators on Manifolds

In this chapter, we focus on the classical theory concerning how to define a pseudodifferential operator of Hörmander class on manifolds, the difficulties arising from this, such as dealing with the action of change of variables, and the techniques developed so far in order to address these questions. We deal with the case of a smooth manifold \mathcal{M} without boundary.

In the first section, we briefly recall the standard method for defining Ψ DOs on manifolds. This relies on the use of local coordinates and the action of change of variables on a Ψ DO in $L_{\rho,\delta}^m(X)$. We observe that the classical results require the condition

$$0 \leq 1 - \rho \leq \delta < \rho \leq 1 \tag{2.1}$$

(see Figure 2.1) in order to have a well posed definition on a manifold. More precisely, we will notice that, unless the condition (2.1) is satisfied the action of change of variables produces undesirable effects. For example, the type of the symbol

associated with a Ψ DO depends on the choice of local coordinates. Condition (2.1) can be relaxed and one can work under the less restrictive assumption

$$0 \leq \delta < \rho \leq 1 \quad \wedge \quad \rho > 1/2, \quad (2.2)$$

(see Figure 2.1). In fact, we can always think to ‘artificially’ increase δ (Recall $S_{\rho,\delta}^m(X \times \mathbb{R}^n) \subseteq S_{\rho,\delta'}^m(X \times \mathbb{R}^n)$). Therefore, requirement (2.2) is essentially the only real constraint necessary in the classical theory.

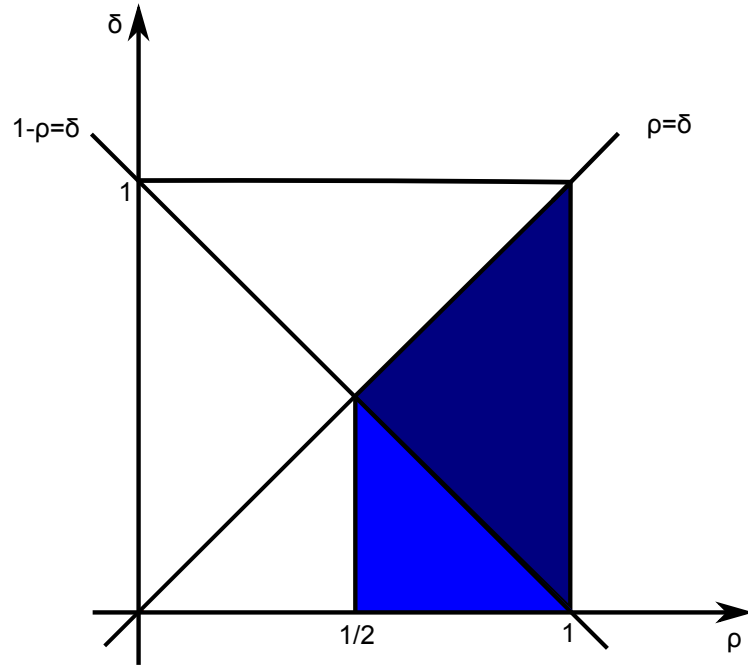


FIGURE 2.1: Condition (2.1) (dark blue) and condition (2.2) (dark blue and blue).

Compared to the symbolic calculus for pseudodifferential operators on \mathbb{R}^n , the symbolic calculus for pseudodifferential operators on manifolds is not as well-established. In the Euclidean setting, one can obtain positive results for a larger range of Ψ DOs than in the manifold setting. In fact, it appears that the classical method to define Ψ DOs on manifolds, based on local coordinates, does not take into full account the underlying geometry and topology of the manifold. In the late

90's, some mathematicians developed further an idea first introduced by Widom in his articles [13, 14]. By using a rather general notion of phase function, he suggested a method for defining full symbols of Ψ DOs on a manifold with a linear connection and constructed a new version of symbolic calculus.

In the second section, we review the existing literature that arises from Widom's idea. The forces driving the efforts of Pflaum, Safarov and Sharafutdinov, among others, to go beyond the classical approach are essentially two.

Primarily, an intrinsic definition of Ψ DO on manifolds. It is evident that the function $\langle x - y, \theta \rangle$ is not invariantly defined on a manifold. An immediate consequence is that the integral representation of the kernel of a Ψ DO

$$\int e^{i\langle x-y, \theta \rangle} \sigma(x, \theta) d\theta, \quad \sigma \in S_{\rho, \delta}^m(X \times \mathbb{R}^n),$$

depends in general on the choice of local coordinates. Therefore, the first step undertaken in order to improve the classical theory was the proposal of a representation which does not depend on the choice of local coordinates. The function $x - y$ was replaced by another function invariantly defined through the use of geodesics and linear connections.

The second idea aims at establishing an extension of the theory of symbolic calculus to cases otherwise excluded. The problems lay essentially in the estimates for the derivatives of symbols since differentiation in x depends once again on the choice of local coordinates. Consequently, the type (ρ, δ) of a Ψ DO, understood in a classical sense, is not preserved under the action of change of variables. Traditionally, if $\delta < 1 - \rho$ (or at least when $\rho \leq 1/2$), one can neither define symbols of a Ψ DO nor obtain most of the other standard results in the usual way. Only Safarov has succeeded in actually improving the range of Ψ DOs for some symbolic calculus

results in [16]. By replacing the differentiation in x in the definition of symbols and using a technique based on integration by parts, he managed to obtain new classes of symbols and extend all the basic results of the classical theory when the less restrictive conditions are satisfied.

Assume once and for all that \mathcal{M} is a smooth n -dimensional manifold without boundaries. Moreover, we denote points of \mathcal{M} by x, y or z and the covectors from $T_x^*\mathcal{M}$, $T_y^*\mathcal{M}$, $T_z^*\mathcal{M}$ by ξ, η, ζ respectively. The same letters denote also coordinates on \mathcal{M} and the corresponding dual coordinates in the fibre of $T^*\mathcal{M}$. We indicate with $\mathcal{C}^\infty(\mathcal{M})$ and $\mathcal{C}_0^\infty(\mathcal{M})$ the space of all smooth complex-valued functions on \mathcal{M} and the subspace of all functions with compact support respectively.

2.1 Classical Theory

In this section, we recall the well-established theory for defining Ψ DOs on manifolds using charts and local coordinates. This is usually referred to as the classical theory of Ψ DO on manifolds. Roughly speaking, this method consists of breaking up the operator over each chart of the manifold and checking that each restriction is a pseudodifferential in the sense of Chapter 1. One can immediately understand that in order to apply this method, we need to investigate what happens to a Ψ DO when a change of variables is performed in the Euclidean setting. Then, all the theory develops in perfect parallelism with the Euclidean case. We will discuss carefully the condition

$$0 \leq 1 - \rho \leq \delta < \rho \leq 1, \quad (2.3)$$

required for the result concerning the action of change of variables over a Ψ DO.

This section is based on [11, Section 1.4], to which we refer the interested reader for a detailed discussion of this topic.

We start by investigating what happens to a Ψ DO $A_1 \in L_{\rho,\delta}^m(X_1)$ when one performs a diffeomorphism $\mu : X_2 \rightarrow X_1$ from one open set $X_2 \subseteq \mathbb{R}^n$ onto another open set $X_1 \subseteq \mathbb{R}^n$. The induced transformation $\mu^* : \mathcal{C}^\infty(X_1) \rightarrow \mathcal{C}^\infty(X_2)$, which maps a function u to the function $u \circ \mu$ is an isomorphism and transform $\mathcal{C}_0^\infty(X_1)$ into $\mathcal{C}_0^\infty(X_2)$. Define $A_2 : \mathcal{C}_0^\infty(X_2) \rightarrow \mathcal{C}^\infty(X_2)$ with the help of the commutative diagram

$$\begin{array}{ccc} \mathcal{C}_0^\infty(X_1) & \xrightarrow{A_1} & \mathcal{C}^\infty(X_1) \\ \mu^* \downarrow & & \downarrow \mu^* \\ \mathcal{C}_0^\infty(X_2) & \xrightarrow{A_2} & \mathcal{C}^\infty(X_2), \end{array} \quad (2.4)$$

that is,

$$A_2 u = [A_1(u \circ \mu^{-1})] \circ \mu. \quad (2.5)$$

Let A_1 be a Ψ DO from the class $L_{\rho,\delta}^m(X_1)$ with integral representation

$$A_1 u(x) = \int \int e^{i\langle x-y, \theta \rangle} a(x, y, \theta) u(y) dy d\theta, \quad a \in S_{\rho,\delta}^m(X_1 \times X_1 \times \mathbb{R}^n).$$

Then

$$A_2 u(x) = \int \int e^{i\langle \mu(x)-z, \theta \rangle} a(\mu(x), z, \theta) u(\mu^{-1}(z)) dz d\theta \quad (2.6)$$

and setting $z = \mu(y)$, we obtain

$$A_2 u(x) = \int \int e^{i\langle \mu(x)-\mu(y), \theta \rangle} a(\mu(x), \mu(y), \theta) |\det J(\mu)(y)| u(y) dy d\theta,$$

where $J(\mu)$ is the Jacobi matrix of the transformation μ . As a consequence of the following theorem we obtain that if $1 - \rho \leq \delta$ the operator A_2 is also a Ψ DO.

Theorem 2.1. *Let $1 - \rho \leq \delta$ and $\mu : X_2 \rightarrow X_1$ is a diffeomorphism. If A_1 is in $L_{\rho,\delta}^{m_1}(X_1)$, then A_2 , defined as in (2.5), is in $L_{\rho,\delta}^{m_1}(X_2)$. If A_1 is a classical Ψ DO, then A_2 is a classical Ψ DO.*

We are interested in understanding better the reason why we need to require the condition $1 - \rho \leq \delta$. For this purpose, we present an outline of the proof of Theorem 2.1.

By Taylor's expansion, there exists a matrix $M(x, y) \in [\mathcal{C}^\infty(X_2 \times X_2)]^{n \times n}$ such that $x - y = M^t(x, y)(\mu(x) - \mu(y))$. Then, we consider the change of variables $\theta = M(x, y)\tilde{\theta}$ so that we obtain

$$\begin{aligned} A_2 u(x) &= \int \int e^{i\langle x-y, \tilde{\theta} \rangle} a(\mu(x), \mu(y), M(x, y)\tilde{\theta}) \\ &\quad \times |\det J(M)(x, y)| |\det J(\mu)(y)| u(y) dy d\tilde{\theta}. \end{aligned}$$

Finally, we notice that $|\det J(M)(x, y)|$ and $|\det J(\mu)(y)|$ are in $S_{1,0}^0(X_2 \times X_2 \times \mathbb{R}^n)$ because they are homogeneous functions of order 0 in $\tilde{\theta}$ (see Example 1.1) and

$$a(\mu(x), \mu(y), M(x, y)\tilde{\theta}) \in S_{\rho,\delta'}^m(X_2 \times X_2 \times \mathbb{R}^n).$$

An intuition of this fact is given by noticing that since $a \in S_{\rho,\delta}^m(X_1 \times X_1 \times \mathbb{R}^n)$, by the chain rule, for any compact $K \subseteq \mathbb{R}^n \times \mathbb{R}^n$ there exists $C = C_K$ such that

$$|\partial_{x_j} a(\mu(x), \mu(y), M(x, y)\tilde{\theta})| \leq C(\langle \tilde{\theta} \rangle^{m+\delta} + \langle \tilde{\theta} \rangle^{m+1-\rho}) \quad \text{for all } (x, y) \in K.$$

Therefore, operators defined via (2.6) in order to be of type (ρ, δ) require the condition $\delta' \leq \delta$, or equivalently $1 - \rho \leq \delta$.

Condition $\rho > \delta$ is instead required in order to have meaningful asymptotic expansions (see Remark 1.16).

We are now ready to define pseudodifferential operators on manifolds. Assume that we are given an open covering $\{\mathcal{C}\}_{j \in J}$ of \mathcal{M} and, for each index j , a homeomorphism μ_j of \mathcal{C}_j onto an open subset of \mathbb{R}^n . A chart in \mathcal{M} is then any pair (\mathcal{C}, μ) consisting of an open subset of \mathcal{M} and a diffeomorphism μ of \mathcal{C} onto an open subset of \mathbb{R}^n .

Definition 2.2. A continuous linear map $A : \mathcal{C}_0^\infty(\mathcal{M}) \rightarrow \mathcal{C}^\infty(\mathcal{M})$ is called a pseudodifferential operator on \mathcal{M} of order m and type (ρ, δ) , and denoted by $L_{\rho, \delta}^m(\mathcal{M})$, if for any chart (\mathcal{C}, μ) in \mathcal{M} the operator $A_{\mathcal{C}}^\mu : \mathcal{C}_0^\infty(\mathcal{C}) \rightarrow \mathcal{C}^\infty(\mathcal{C})$ given by $u \rightarrow [A(u \circ \mu^{-1})] \circ \mu$ is in $L_{\rho, \delta}^m(\mu(\mathcal{C}))$.

Furthermore, one can prove that the class of symbols (amplitudes) defined on the cotangent bundle $T^*\mathcal{M}$ ($T^*\mathcal{M} \times \mathcal{M}$), denoted by $S_{\rho, \delta}^m(T^*\mathcal{M})$ ($S_{\rho, \delta}^m(T^*\mathcal{M} \times \mathcal{M})$), as well as the class of operators $L_{\rho, \delta}^m(\mathcal{M})$, are well defined under the action of change of coordinates for $1 - \rho \leq \delta$. The discussion regarding the principal symbol principal symbol of class $S_{\rho, \delta}^m(T^*\mathcal{M})$ deserves special care. In fact, we defined principal symbols only in the case of classical symbols (see Definition 1.17). In this case, they are uniquely defined because of the special asymptotic expansion consisting of homogeneous functions, which describes them. However, we can give a notion of the principal symbol for the class $S_{\rho, \delta}^m(X)$ and more generally for $S_{\rho, \delta}^m(T^*\mathcal{M})$. One can think to define this principal symbol as the first term of the asymptotic expansion (1.9). However, in this case the definition of the principal symbol would not be unique any more because one may add terms of lower orders and rearrange

the asymptotic expansion (1.9). In order to make it unique, we need to regard the principal symbol as an equivalence class defined modulo terms of lower order.

Lemma 2.3. *Given a diffeomorphism $\mu : X_2 \rightarrow X_1$ and a properly supported ΨDO $A \in L_{\rho,\delta}^m(X_1)$ with $1 - \rho \leq \delta$. Let A_2 be determined by (2.6). Then*

$$\sigma_{A_2}(x, \xi) = \sigma_{A_1}(\mu(x), [J^t(\mu(x))]^{-1}\xi) \mod S_{\rho,\delta}^{m-2(\rho-1/2)}(X_2 \times \mathbb{R}^n).$$

The proof of this lemma can be found in [11, Section 4].

Lemma 2.3 shows that modulo symbols of lower order, the symbol of all operators obtained from A by change of variables form a well defined function on the cotangent bundle T^*X_1 .

Also, in view of Theorem 2.1 the class of classical ΨDO s is well defined on \mathcal{M} . If A is a classical ΨDO on \mathcal{M} , then the principal symbol can be considered as a homogeneous function on $T^*\mathcal{M}$ with degree of homogeneity equal to m , since two functions $a_1(x, \xi)$ and $a_2(x, \xi)$, positively homogeneous in ξ for $|\xi| \geq 1$ of degree m , which define the same equivalence class in $S^m(X \times \mathbb{R}^n)$ modulo $S^{m-1}(X \times \mathbb{R}^n)$, must coincide for $|\xi| \geq 1$.

Similarly to the Euclidean case, by continuity and duality one can extend the definition of a ΨDO s $A : \mathcal{C}_0^\infty(\mathcal{M}) \rightarrow \mathcal{C}^\infty(\mathcal{M})$ to the class of distributions compactly supported on a manifold. Namely, $\mathcal{E}'(\mathcal{M}) \rightarrow \mathcal{D}'(\mathcal{M})$. Moreover, one can use the notion of properly supported ΨDO s to prove a result analogous to Proposition 1.11.

Theorem 2.1 is a corollary of a more general result, which is relevant for our work. In order to state this result, we need first to introduce the following class of functions.

Definition 2.4. We say that a real valued function $\varphi \in \mathcal{C}^\infty(X \times X \times \mathbb{R}^n \setminus \{0\})$ is a special phase function if φ is positively homogeneous of degree 1 in θ and

- $\nabla_\theta \varphi(x, y, \theta) = 0$ if and only if $x = y$;
- $\nabla_x \varphi(x, y, \theta) \Big|_{y=x} = \theta$.

The conditions above for nearby x and y are equivalent to the following single condition:

$$\varphi(x, y, \theta) = \langle x - y, \theta \rangle + \mathcal{O}(|x - y|^2 |\theta|), \quad (2.7)$$

where $\mathcal{O}(|x - y|^2 |\theta|)$ is a factor positively homogeneous of degree 1 in θ and with a zero of second order at $y = x$. In fact, by Taylor's formula for fixed y , we have

$$\varphi(x, y, \theta) = \varphi(y, y, \theta) + \langle \nabla_x \varphi(x, y, \theta) \Big|_{x=y}, x - y \rangle + \mathcal{O}(|x - y|^2 |\theta|). \quad (2.8)$$

So we only need to notice that by Euler's identity

$$\varphi(x, y, \theta) = \langle \theta, \nabla_\theta \varphi(x, y, \theta) \rangle, \quad \forall \theta \in \mathbb{R}^n \setminus \{0\}.$$

Notice that a special phase function φ is a phase function in the sense of Definition 1.2. Consequently, the integral

$$\int \int e^{i\varphi(x, y, \theta)} a(x, y, \theta) u(x) dy d\theta, \quad \text{for all } a \in S_{\rho, \delta}^m(X \times X \times \mathbb{R}^n) \quad (2.9)$$

is a well defined oscillatory integral for any special phase function φ . Now we propose a proof of the fact that, assuming condition (2.1), oscillatory integrals of the form (2.9) are indeed pseudodifferential operators. We will denote by $L_{\rho, \delta}^m(\varphi, X)$ the class of operators defined by (2.9). Notice that the class $L_{\rho, \delta}^m(\langle x - y, \xi \rangle, X)$

coincides with the class $L_{\rho,\delta}^m(X)$ of pseudodifferential operators defined in Definition 1.6. The following theorem and its proof are taken from [11, Section 19].

Theorem 2.5. *If $1 - \rho \leq \delta$ and $\varphi(x, y, \theta)$ is a special phase function, then $L_{\rho,\delta}^m(\varphi, X)$ coincides with the class of pseudodifferential operators $L_{\rho,\delta}^m(X)$.*

Proof. Let φ, φ_1 be two special phase functions. Clearly, it suffices to verify the inclusion

$$L_{\rho,\delta}^m(\varphi, X) \subseteq L_{\rho,\delta}^m(\varphi_1, X). \quad (2.10)$$

First, we claim that for nearby x and y , the difference $\varphi_1 - \varphi$ can be written in the form

$$\varphi_1 - \varphi = \sum_{j,k=1}^n b_{jk} \partial_{\theta_j} \varphi \partial_{\theta_k} \varphi \quad (2.11)$$

where b_{jk} is positively homogeneous of degree 1 in θ . In fact,

$$\frac{\partial \varphi}{\partial \theta_j} = (x_j - y_j) + \sum_{k=1}^n a_{jk}(x_k - y_k),$$

where a_{jk} are positively homogeneous of degree 0 in θ and $a(x, x, \theta) = 0$. This can also be written in the form

$$\nabla_{\theta} \varphi = (I + A)(x - y),$$

where I is the unit matrix and A is a matrix with elements positively homogeneous of degree 0 in θ , equal to 0 for $x = y$. Therefore for nearby x and y the matrix $(I + A)^{-1}$ exists and has elements positively homogeneous of degree 0 in θ . This means that we may write

$$x_j - y_j = \sum_{k=1}^n \tilde{a}_{jk} \partial_{\theta_k} \varphi, \quad (2.12)$$

where \tilde{a}_{jk} are positively homogeneous of degree 0 in θ .

Now, using (2.7), we observe that $\varphi_1 - \varphi$ has a zero of order two on the diagonal $\mathcal{D} = \{(x, y) \in X \times X : x = y\}$ and by Taylor's formula

$$\varphi_1 - \varphi = \sum_{j,k=1}^n \tilde{b}_{jk}(x_j - y_j)(x_k - y_k), \quad (2.13)$$

where \tilde{b}_{jk} are positively homogeneous of degree 1 in θ .

In order obtain (2.11), we just need to put together (2.12) and (2.13).

Now, consider the homotopy

$$\varphi_t = \varphi + t(\varphi_1 - \varphi), \quad 0 \leq t \leq 1.$$

Each of the functions φ_t is a special phase function. A trivial repetition of the above argument shows that (2.11) can be turned into

$$\varphi_1 - \varphi = \sum_{j,k=1}^n b_{jk}^{(t)} \partial_{\theta_j} \varphi_t \partial_{\theta_k} \varphi_t,$$

where $b_{jk}^{(t)}$ are positively homogeneous of degree 1 in θ and depends smoothly on t .

Now, let P_t be the oscillatory integral given by the formula

$$P_t u(x) = \int \int e^{i\varphi_t(x,y,\theta)} p(x, y, \theta) u(y) dy d\theta, \quad p \in S_{\rho,\delta}^m(X \times X \times \mathbb{R}^n).$$

Then

$$\begin{aligned} \frac{d^r}{dt^r}(P_t u) &= \int \int e^{i\varphi_t(x,y,\theta)} i^r (\varphi_1 - \varphi)^r p(x, y, \theta) u(y) dy d\theta \\ &= \sum_{j_1, \dots, j_{2r}=1}^n \int \int e^{i\varphi_t(x,y,\theta)} d_{j_1, \dots, j_{2r}}^{(t)} \partial_{\theta_{j_1}} \varphi_t \dots \partial_{\theta_{j_{2r}}} \varphi_t p(x, y, \theta) u(y) dy d\theta, \end{aligned} \quad (2.14)$$

where $d_{j_1, \dots, j_{2r}}^{(t)}$ are positively homogeneous of degree r in θ . Without loss of generality, we may assume that $p(x, y, \theta) = 0$ for $|\theta| \leq 1$ (see Remark 1.7), so that $d_{j_1, \dots, j_{2r}}^{(t)} p \in S_{\rho, \delta}^{m+r}(X)$ uniformly in t . Integrating by parts in (2.14), using the formula

$$e^{i\varphi_t} \partial_{\theta_j} \varphi_t = i^{-1} \partial_{\theta_j} e^{i\varphi_t}.$$

By means of this integration and condition $1 - \rho \leq \delta$, it follows that

$$\frac{d^r}{dt^r} P_t \in L_{\rho, \delta}^{m+r(1-2\rho)}(\varphi_t),$$

where all estimates are uniform in t . Now, put

$$Q_j = \frac{(-1)^j}{j!} \frac{d^j P_t}{dt^j} \Big|_{t=1} \in L_{\rho, \delta}^{m+j(1-2\rho)}(\varphi_t).$$

By Taylor's formula,

$$P_t = \sum_{j=0}^{k-1} \frac{(t-1)^j}{j!} \frac{d^j P_t}{dt^j} \Big|_{t=1} + (t-1)^k \int_0^1 \frac{s^{k-1}}{(k-1)!} \frac{d^k P_s}{ds^k} ds.$$

Therefore, for $t = 0$,

$$P_0 = \sum_{j=0}^{k-1} Q_j + (-1)^k \int_0^1 \frac{s^{k-1}}{(k-1)!} \frac{d^k P_s}{ds^k} ds.$$

The remainder has a kernel with increasing smoothness as $k \rightarrow +\infty$. It is therefore clear that if $Q \sim \sum_{j=0}^{\infty} Q_j$ (adding the amplitudes asymptotically), then

$$Q \in L_{\rho, \delta}^m(\varphi_1, X) \quad \text{and} \quad P_0 - Q \text{ has smooth kernel,}$$

which shows (2.10). □

Remark 2.6. It is clear from the construction that classical amplitudes remain classical under this procedure. Moreover, Theorem 2.5 implies Theorem 2.1.

Similarly to what we did in Section 1.1 to regularise oscillatory integrals, we use a method based on integration by parts to show that factors of the form $x - y$ can be used to decrease the order. An analogous strategy is used for proving Proposition 1.9 part d).

2.2 Modern Literature

The method presented in section 2.1 to define Ψ DOs on a smooth manifold without boundary is based on the use of local coordinate systems. In particular, the condition

$$0 \leq 1 - \rho \leq \delta < \rho \leq 1 \quad (2.15)$$

is required in order for the classes of symbols $S_{\rho,\delta}^m(X \times X \times \mathbb{R}^n)$ to be invariantly defined under the action of change of variables. Ideally we would like to lift this requirement or at least relax it.

In the classical theory symbols on manifolds are well defined only in their principal part. In 1978, Widom suggested a new method method for defining the full symbol of a Ψ DO on a manifold endowed with a linear connection Γ . He constructed a new version of symbolic calculus (see [13, 14]). The fact that a linear connection is a global object enables one to associate with a Ψ DO its full symbol, which is a function over the cotangent bundle $T^*\mathcal{M}$ depending on the choice of the linear connection. In addition he manages to improve results in applications such as functional calculus. The main disadvantage of this technique is the absence of an explicit formula representing the Schwartz kernel of a Ψ DO via its symbol. As

a consequence, one has to assume that Ψ DOs and the corresponding classes of amplitudes are defined in local coordinates, which makes it impossible to extend the definition to the case $\rho \leq 1/2$. More advanced results in this direction were obtained later in [20] by Fulling and Kennedy. As mentioned in [20, 21], Drager introduced similar phase functions in [22].

In 1996, Pflaum (see [12, 15]) and Safarov (see [16]) developed a theory for the symbolic calculus of Ψ DOs using invariant oscillatory integrals over the cotangent bundle $T^*\mathcal{M}$. This technique is still based on the geometric approach first introduced by Widom. Roughly speaking, they replaced the standard phase function $\langle x - y, \xi \rangle$ of a Ψ DO, which is obviously not invariantly defined on $T^*\mathcal{M} \times \mathcal{M}$, by the global phase function

$$\varphi_\tau(x, y, \zeta_\tau) = -\langle \dot{\gamma}_{x,y}(\tau), \zeta_\tau \rangle,$$

where $\gamma_{x,y} : [0, 1] \rightarrow \mathcal{M}$ is the geodesic such that $\gamma_{x,y}(0) = x$ and $\gamma_{x,y}(1) = y$, for x and y sufficiently close, and $\zeta_\tau \in T_{z_\tau}^*\mathcal{M}$, with $z_\tau = \gamma_{x,y}(\tau)$. Notice that when $\tau = 0$ and $\tau = 1$, then $(z_\tau, \zeta_\tau) = (x, \xi)$ and $(z_\tau, \zeta_\tau) = (y, \eta)$ respectively.

In [15], Pflaum defined the kernel of a Ψ DO in the space of functions by the formula

$$\mathcal{A}_0(x, y) = \chi(x, y) \int_{T_x^*\mathcal{M}} e^{i\varphi_0(x, \xi; y)} a(x, \xi) d\xi, \quad (2.16)$$

where $a(x, \xi)$ is a function on $T^*\mathcal{M}$ of class $S_{\rho, \delta}^m(T^*\mathcal{M})$, $\{y^k\}$ are normal coordinates centred at x and χ is a smooth cut-off function vanishing outside a neighbourhood of the diagonal. He obtained asymptotic expansions for the symbols of the adjoint operator and the composition of Ψ DOs. Later, he extended his results by considering the more general case obtained by replacing φ_0 by φ_τ and dealing with τ -symbols

(see [12]). However, Pflaum proves his results with the use of local coordinates, hence he had to assume that $1 - \rho \leq \delta$, as in the classical theory.

An integral of the form (2.16) leads to a natural question regarding the intrinsic geometric meaning of it. More precisely, when considering the action of the kernel $\mathcal{A}(x, y)$ over a function $u \in \mathcal{C}_0^\infty(\mathcal{M})$, we have

$$Au(x) = \int_{\mathcal{M}} \mathcal{A}_0(x, y)u(y)dy.$$

Obviously there is an issue regarding the meaning of such integrals and measure dy . Pflaum decided to deal with it using local coordinates. He fixes local coordinates for x and uses normal local coordinates centred at x for variable y .

In [16], Safarov managed to avoid the use of local coordinates. In fact, the integral

$$\int_{T^*\mathcal{M}} e^{i\varphi_1(x;y,\eta)} a(y, \eta)u(y)dy d\eta.$$

is well defined because the measure $d\eta dy$ has an intrinsic geometric meaning on $T^*\mathcal{M}$ and therefore there is hope to have a coordinate-free definition. However, we notice that we are dealing with a situation where the symbol corresponds to the dual symbol according to the case of local coordinates. If instead we want to deal with a situation closer to the one proposed in Chapter 1, we need to consider the operator of the form

$$\int_{T^*\mathcal{M}} e^{i\varphi_0(x,\xi;y)} a(x, \xi)u(y)dy d\xi,$$

generated by the kernel $\mathcal{A}_0(x, y)$. We observe immediately that this integral does not behave as a function but as a 1-density in the variable x . In addition, the kernel $\mathcal{A}_0(x, y)$ is a function in y and therefore integration in this variable depends on the choice of local coordinates.

Safarov addressed these issues by introducing a weight factor in the integral defining the kernel:

$$\mathcal{A}_\tau(x, y) = p_{\varkappa, \tau}(x, y) \int_{T_{z_\tau}^* \mathcal{M}} e^{i\varphi_\tau(x, y, \zeta_\tau)} \sigma_A(z_\tau, \zeta_\tau) d\zeta_\tau. \quad (2.17)$$

He considered the general case of an operator mapping a \varkappa -density into a \varkappa -density. The weight factor in this case $p_{\varkappa, \tau}(x, y)$ is consequently a $(\varkappa - 1)$ -density in x and a $(1 - \varkappa)$ -density in y . Finally, since 0-densities are functions, the standard case is just a special case of this more general theory. We say more about densities in the next chapter.

The most significant advantage of the approach used by Safarov is the fact that even though all calculations are carried out in local coordinates, the results does not depend on the choice of coordinates and obtain global results. This leads to

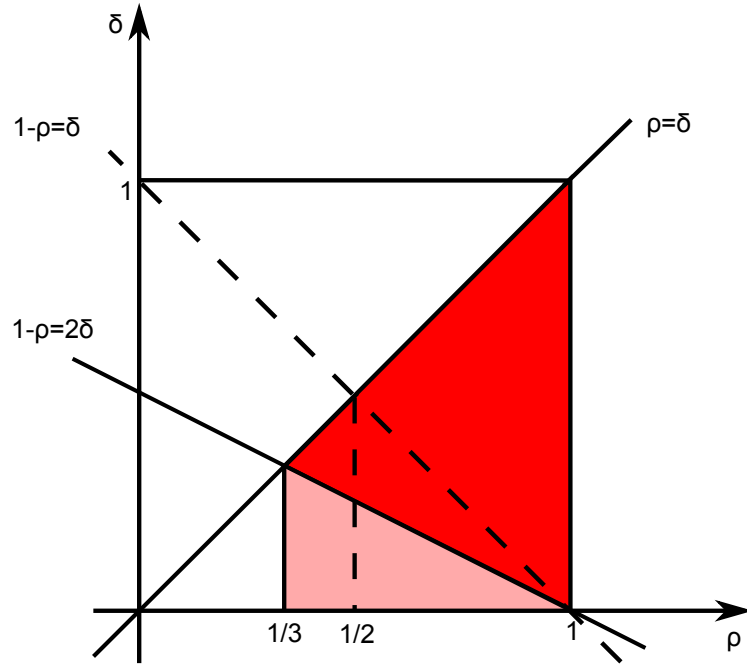


FIGURE 2.2: Condition (2.18) (red and pink) and condition (2.19) (red), the dashed lines represent conditions (2.1) and (2.2).

the relaxation the condition (2.15) and working under either the condition

$$0 \leq \delta < \rho \leq 1 \quad \wedge \quad \rho > 1/3, \quad (2.18)$$

or at least

$$0 \leq \delta < \rho \leq 1 \quad \wedge \quad 1 - \rho \leq 2\delta, \quad (2.19)$$

(see Figure 2.2) when the linear connection considered is symmetric or equivalently the torsion tensor of the connection is identically zero.

He also introduces a new class of symbols $S_{\rho,\delta}^m(\Gamma, T^*\mathcal{M})$; in particular the symbol σ_A in (2.17) is an element of it. The definition of this class requires the use of the horizontal lifts ∇_j of the vector fields ∂_{x^j} . The definition of $S_{\rho,\delta}^m(\Gamma, T^*\mathcal{M})$ coincides with the one for $S_{\rho,\delta}^m$ on Euclidean spaces, except for the fact that the usual differentiations ∂_{x_j} are replaced by ∇_j . Safarov proves that the class $S_{\rho,\delta}^m(\Gamma, T^*\mathcal{M})$ does not actually depends on the connection Γ under the classical assumption $1 - \rho \leq \delta$.

A formula for compositions and adjoints of Ψ DOs on manifolds, analogous to the one on \mathbb{R}^n described in Proposition 1.18, is proved. Namely, for a manifold endowed with a symmetric connection, under condition (2.18) one has

$$\sigma_{AB}(x, \xi) \sim \sum_{\alpha, \beta, \gamma} \frac{1}{\alpha!} \frac{1}{\beta!} \frac{1}{\gamma!} P_{\beta, \gamma}^{(\varkappa)}(x, \xi) D_\xi^{\alpha+\beta} \sigma_A(x, \xi) D_\xi^\gamma \nabla_x^\alpha \sigma_B(x, \xi), \quad (2.20)$$

when the type of σ_A and σ_B satisfy (2.18), $P_{\beta, \gamma}^{(\varkappa)}$ are some weight factors and $\nabla_x^\alpha = \frac{1}{q!} \sum \nabla_{j_1} \dots \nabla_{j_q}$, with $q = |\alpha|$ and the sum taken over all ordered collections of indices i_1, \dots, i_q corresponding to the multi-index α . In addition, one has

$$\sigma_{A^*}(x, \xi) \sim \sum_{\alpha} \frac{1}{\alpha!} D_\xi^\alpha \overline{\nabla_x^\alpha \sigma_A(x, \xi)}, \quad (2.21)$$

when the type of σ_A satisfies only the condition $0 \leq \delta < \rho \leq 1$. Up to minor adjustments, we observe that the formula is very similar to the one for the Euclidean case. The asymptotic expansion for the composition of Ψ DOs is an important result because it lays the foundations for some important applications, such as boundedness on Sobolev spaces and functional calculus of first order differential operators.

Applying the composition formula (2.20), one can show that

$$\sigma_A(x, \xi) = \sigma_A(z_\tau, \zeta_\tau) \Big|_{\tau=0} \sim \sum_{\alpha} P_{\alpha}(x) \partial_{\xi}^{\alpha} a(x, \xi), \quad (2.22)$$

where $a(x, \xi)$ is the symbol appearing in (2.16) and P_{α} are components of some tensor fields. Using (2.22), one can rewrite all the results obtained in [16] in terms of the slightly different notion of symbols by Pflaum. In particular, his composition formula can be written in the form (2.20) with some other polynomials $\tilde{P}_{\beta, \gamma}^{(\varkappa)}$.

Sharafutdinov also obtained in [18] a composition formula for Ψ DOs on manifolds. He considered the less general case of operators acting in the space of functions ($\varkappa = 0$) and $\tau = 0$ ($z_\tau = x$). He chose to give a direct proof instead of deducing the formula from (2.20) and (2.22). Moreover, he considered only the classes $\Psi_{1,0}^m$. Because of these additional assumptions, Sharafutdinov succeed in giving an alternative description of the polynomials $\tilde{P}_{\beta, \gamma}^{(0)}$ that may be useful for obtaining more explicit composition formulae (this investigation was continued in [23]). He also obtained an asymptotic expansion for σ_{A^*} in the case $\varkappa = 1/2$ and $\tau = 0$.

In [12, 14, 19, 20] the authors considered Ψ DOs acting between spaces of sections of vector bundles over \mathcal{M} . In this case, in order to construct a global symbolic calculus, it is sufficient to define the parallel displacement and horizontal curves in the induced bundles over $T^*\mathcal{M}$. This can be achieved by introducing linear

connections on \mathcal{M} and the vector bundles over \mathcal{M} . After that the results are stated and proved in the same way as in the scalar case. All the papers mentioned above deal only with symbols whose restriction to compact subsets K of \mathcal{M} belong to $S_{\rho,\delta}^m(K)$ with $1 - \rho \leq \delta$.

Chapter 3

Invariant Approach

In this chapter, we present a new approach to construct an invariant symbolic calculus for Ψ DOs of Hörmander type on smooth manifolds. This new approach and the results deriving from it reinforce the evidence, already revealed in earlier works, that the key for a more comprehensive theory of symbolic calculus for Ψ DO lies in an appropriate notion of phase function.

Compared to previous literature on the topic, the most considerable element of innovation in this work is the use of a notion of phase function which is non-linear in the phase variable and fully analytical. Our approach does not rely on the use of linear connections. Instead, the novelty of the work presented is to develop idea suggested by the and Safarov. In [17, Section 7.8], they notice that

‘One does not need a linear connection or even an exponential map to define Ψ DOs on a manifold in a coordinate-free manner. It is sufficient to fix a globally defined phase function satisfying certain conditions.’

Namely, we choose a suitable function $\varphi(x, \xi; y)$ on $T^*\mathcal{M} \times \mathcal{M}$. If $a(x, \xi; y) \in \mathcal{C}^\infty(T^*\mathcal{M} \times \mathcal{M})$ such that a is an amplitude in any local coordinate system, then

$$\mathcal{A}_\varphi(x, y) = \int e^{i\varphi(x, \xi; y)} a(x, \xi; y) d\xi \quad (3.1)$$

is the Schwartz kernel of a Ψ DO acting in the space of 1-densities.

In Section 2.2, we noticed that Safarov's work might be considered to a certain extent the most advanced in terms of results for the theory of symbolic calculus on manifolds. In addition to proposing a comprehensive discussion regarding Ψ DOs acting in the space of densities, his composition formula and functional calculus are invariant and remain still unbeaten in terms of conditions required on the type (ρ, δ) of the operators for which they are developed.

In the first section of this chapter, we introduce formally the notion of global phase functions that we will use to define a new version of symbolic calculus on manifolds as proposed by Safarov and McKeag. After showing some basic properties, we introduce the related concept of the curved derivative of a smooth function on $T^*\mathcal{M}$ along a surface suitably generated from the global phase function. This curved derivative has a correspondence in the theory developed by Safarov with the operator ∇_x^α mentioned in the previous chapter and, likewise, takes into account the underlying geometry and topology of the manifold.

Section 3.2 reviews the definition of the Hörmander class of symbols and the related properties in light of the new approach. The concept of symbols is intertwined with the one of global phase functions through the curved derivative introduced in Section 3.1. We prove that, under the classical condition $1 - \rho \leq \delta$, the definition does not actually depend on the choice of the global phase function. Instead, when $1 - \rho > \delta$, we obtain a new class of symbols. In local coordinates, these symbols

belong to $S_{\rho,1-\rho}^m$. All the basic results on symbols remain valid under the conditions

$$0 \leq \delta < \rho \leq 1 \quad \wedge \quad 1 - \rho \leq 2\delta. \quad (3.2)$$

Since we can always artificially increase the parameter δ (recall $S_{\rho,\delta_1}^m \subseteq S_{\rho,\delta_2}^m$ for $\delta_1 \leq \delta_2$), conditions (3.2) are equivalent to (2.18) used by Safarov in [16].

Finally, in the third section, we show that an operator with Schwartz kernel as in (3.1) is a Ψ DO. We choose to deal with operators acting on the space of $1/2$ -densities. Two reasons motivate this decision. First, we can introduce the Hilbert space $L^2(\mathcal{M}, \Omega^{1/2})$ of $1/2$ -densities without the use of local coordinates. Second, the adjoint of a given operator acts in the same space. For this reason, we introduce a suitable weight factor in the formula for the kernel.

From now on, we assume \mathcal{M} to be a closed manifold, i.e. a compact manifold without boundary. Simple applications for Ψ DOs on manifolds, such as functional calculus, require this condition. Therefore, compactness is normally assumed and allows us to simplify the notation. Some definitions and results presented in this and the following chapters can be easily extended to the non-compact case.

We often consider functions defined over $T^*\mathcal{M} \times \mathcal{M}$. We use the notation $(x, \xi; y)$ rather than (x, ξ, y) in order to emphasise the independence between the covector $(x, \xi) \in T^*\mathcal{M}$ and the variable $y \in \mathcal{M}$.

3.1 Global Phase Functions

First of all, we introduce the concept of the global phase function.

We denote by $T^*\mathcal{M} \setminus \{0\}$ the space obtained from the cotangent bundle $T^*\mathcal{M}$ by removing its zero section, e.g. $\{(x, \xi) \in T^*\mathcal{M} : \xi \neq 0\}$. We recall that this space is invariant under the action of change of coordinates. We fix a positive function $w(x, \xi) \in \mathcal{C}^\infty(T^*\mathcal{M} \setminus \{0\})$ positively homogeneous in ξ of degree 1 and define

$$\langle \xi \rangle_x = (1 + w^2(x, \xi))^{1/2}.$$

Remark 3.1. All further definitions and results are independent of the choice of w .

Definition 3.2. We say that a real valued function $\varphi(x, \xi; y) \in \mathcal{C}^\infty(T^*\mathcal{M} \setminus \{0\} \times \mathcal{M})$ is a global phase function on \mathcal{M} if φ is positively homogeneous of degree 1 in ξ and

$$- \nabla_\xi \varphi(x, \xi; y) = 0 \text{ if and only if } y = x; \quad (3.3)$$

$$- \nabla_x \varphi(x, \xi; y) \Big|_{y=x} = \xi. \quad (3.4)$$

Remark 3.3. This definition reviews the concept of special phase function introduced in Section 2.1 in the Euclidean context. In fact, we observe that all the properties for a special phase function do not depend on the choice of local coordinates, such as property (3.3) or at least are invariant under the action of change of variables, such as property (3.4). More precisely for (3.4), let $\mu : X_2 \rightarrow X_1$ between two open sets X_1, X_2 in \mathbb{R}^n . Consider $\varphi \in \mathcal{C}^\infty(T^*X_2 \setminus \{0\} \times X_2)$, then one has $\varphi(x, \xi; y) = \varphi(\mu(x), [J^t(\mu(x))]^{-1}\xi; y)$. Consequently,

$$\begin{aligned} & \nabla_x \varphi(\mu(x), [J^t(\mu(x))]^{-1}\xi; y) \Big|_{y=\mu(x)} \\ &= \underbrace{J^t(\mu(x)) (\nabla_x \varphi)(\mu(x), [J^t(\mu(x))]^{-1}\xi; y) \Big|_{y=\mu(x)}}_{=[J^t(\mu(x))]^{-1}\xi \text{ by (3.4)}} \\ & \quad + \underbrace{A(x, \xi) (\nabla_\xi \varphi)(\mu(x), [J^t(\mu(x))]^{-1}\xi; y) \Big|_{y=\mu(x)}}_{=0 \text{ by (3.3)}} = \xi, \end{aligned}$$

for some $A(x, \xi) = [a_{jk}(x, \xi)]_{j,k=1}^n$, with $a_{jk}(x, \xi) \in \mathcal{C}^\infty(T^*X_2)$.

From now on, unless otherwise stated, we assume that φ is a global phase function arbitrarily chosen.

We observe two properties of global phase functions that will be useful later. First, assume $U \times U \subseteq \mathcal{M} \times \mathcal{M}$ is a coordinate patch and $\{y^k\}$ are the same coordinates as $\{x^k\}$. For x and y nearby, we may express conditions above in the following single condition

$$\varphi(x, \xi; y) = \langle x - y, \xi \rangle + \mathcal{O}(|x - y|^2 |\xi|), \quad (x, y) \in U \times U. \quad (3.5)$$

Obviously, this property follows immediately from (2.7) observed for special phase functions. Second,

$$\nabla_y \varphi(x, \xi; y) \Big|_{y=x} = -\xi. \quad (3.6)$$

In fact, $\varphi(x, \xi; x) = 0$ for any $x \in \mathcal{M}$ and therefore

$$0 = \nabla_x \varphi(x, \xi; x) = \nabla_x \varphi(x, \xi; y) \Big|_{y=x} + \nabla_y \varphi(x, \xi; y) \Big|_{y=x} = \xi + \nabla_y \varphi(x, \xi; y) \Big|_{y=x}.$$

In order to introduce the operator $\mathfrak{D}_{\varphi, x}^\beta$ we only need the notion of global phase function introduced previously.

Definition 3.4. Let $a \in \mathcal{C}^\infty(T^*\mathcal{M})$. For any multi-index β and $(x, \xi) \in T^*\mathcal{M}$ fixed, define the operator $\mathfrak{D}_{\varphi, x}^\beta$ by

$$\mathfrak{D}_{\varphi, x}^\beta a(x, \xi) = \frac{d^\beta}{dy^\beta} a(y, -\nabla_y \varphi(x, \xi; y)) \Big|_{y=x}. \quad (3.7)$$

Notice that the left hand-side does not contain the coordinates $\{y^k\}$ even though it depends on it. In order to avoid confusion, we assume that the coordinates $\{y^k\}$ are the same as $\{x^k\}$.

We denote by $\nabla_{\varphi,x}$ the operator defined by

$$\nabla_{\varphi,x}a(x,\xi) = \nabla_x a(x,\xi) - H_y \varphi(x,\xi;y) \Big|_{y=x} \nabla_\xi a(x,\xi), \quad (3.8)$$

where $H_y \varphi(x,\xi;y) = [\partial_{y_k} \partial_{y_j} \varphi(x,\xi;y)]_{k,j=1}^n$.

The operator $\nabla_{\varphi,x}$ has an evident geometrical interpretation. It is the gradient of the function a constrained to the hypersurface $\{(y,\eta) \in T^*\mathcal{M} : y \in \mathcal{M} \text{ and } \eta = -\nabla_y \varphi(x,\xi;y) \in T_y^*\mathcal{M}\}$ for a fixed $(x,\xi) \in T^*\mathcal{M}$ (see Figure 3.1). The reason for

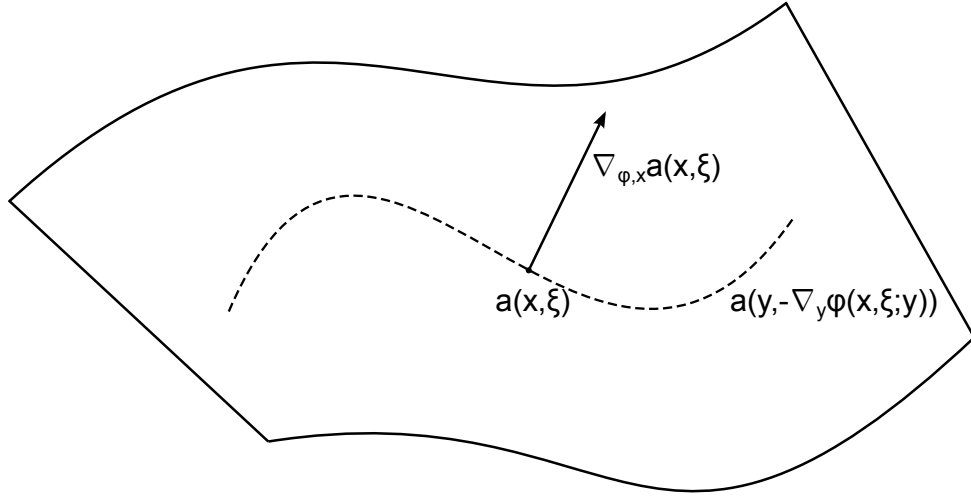


FIGURE 3.1: The operator $\nabla_{\varphi,x}$.

the choice of such operator will become clear in Section 5.2.

Notice that, for fixed $(x,\xi) \in T^*\mathcal{M}$,

$$(y, -\nabla_y \varphi(x,\xi;y)) = (y, \xi + \mathcal{O}(|x-y||\xi|)), \quad y \in \mathcal{M}.$$

Therefore, $\mathfrak{D}_{\varphi,x}^\beta a(x, \xi)$ coincides with $\partial_x^\beta a(x, \xi)$ up to an error term.

3.2 φ -symbols

In this section, we introduce a new class of symbols whose definition depends on the choice of a global phase function φ . The definition is obtained formally by replacing the derivative ∂_x^β used to define the classical symbols $S_{\rho,\delta}^m(X \times \mathbb{R}^n)$ or $S_{\rho,\delta}^m(T^*\mathcal{M})$ (see Definition 1.5) by the operator $\mathfrak{D}_{\varphi,x}^\beta$ introduced in Section 3.1. We call this new class of functions the class of φ -symbols. We will notice that these classes of symbols do not actually depend on the choice of φ when $1 - \rho \leq \delta$ and coincide with the classical ones. On the other hand, when $1 - \rho > \delta$ they give origin to a new class of symbols.

In addition, we review some results established for the classical theory on \mathbb{R}^n . Interestingly enough, in new setting introduced it turns out that we need to only require the condition $1 - \rho \leq 2\delta$. As noted in the introduction of this chapter, this requirement produces $\rho > 1/3$ as the ultimate condition for developing symbolic calculus for Ψ DOs.

Definition 3.5. We denote by $\mathcal{S}_{\rho,\delta}^m(\varphi, T^*\mathcal{M})$ the class of functions $a \in \mathcal{C}^\infty(T^*\mathcal{M})$ such that in any coordinate system $\{x^k\}$ for all α and β there exists a positive constant $C = C_{\mathcal{M},\alpha,\beta}$ such that

$$|\partial_\xi^\alpha \mathfrak{D}_{\varphi,x}^\beta a(x, \xi)| \leq C \langle \xi \rangle_x^{m-\rho|\alpha|+\delta|\beta|}.$$

We say that a function of this class is a φ -symbol of order m and type (ρ, δ) .

Analogously one can define the set of φ -amplitudes as the class of functions $a \in \mathcal{C}^\infty(T^*\mathcal{M} \times \mathcal{M})$ such that in any coordinate system $\{x^k\} = \{y^k\}$ for all α, β and γ there exists a positive constant $C = C_{\mathcal{M}, \alpha, \beta, \gamma}$ such that

$$|\partial_y^\gamma \partial_\xi^\alpha \mathfrak{D}_{\varphi, x}^\beta a(x, \xi; y)| \leq C \langle \xi \rangle_x^{m - \rho|\alpha| + \delta(|\beta| + |\gamma|)}.$$

We denote by $\mathcal{S}_{\rho, \delta}^m(\varphi, T^*\mathcal{M} \times \mathcal{M})$ the class of these elements. Obviously the set of φ -symbols is a subset of the class of amplitudes.

As a consequence of the linearity of differentiation and of the chain rule for differentiation, we obtain the following proposition, which states equivalent rules for φ -amplitudes to the ones for amplitudes in local coordinates.

Proposition 3.6. *If $a \in \mathcal{S}_{\rho, \delta}^{m_1}(\varphi, T^*\mathcal{M} \times \mathcal{M})$ and $b \in \mathcal{S}_{\rho, \delta}^{m_2}(\varphi, T^*\mathcal{M} \times \mathcal{M})$, then*

$$a + b \in \mathcal{S}_{\rho, \delta}^{\max\{m_1, m_2\}}(\varphi, T^*\mathcal{M} \times \mathcal{M}), \quad (3.9)$$

$$ab \in \mathcal{S}_{\rho, \delta}^{m_1 + m_2}(\varphi, T^*\mathcal{M} \times \mathcal{M}). \quad (3.10)$$

Moreover, for $m_1 \leq m_2, \rho_1 \geq \rho_2$ and $\delta_1 \leq \delta_2$

$$\mathcal{S}_{\rho_1, \delta_1}^{m_1}(\varphi, T^*\mathcal{M} \times \mathcal{M}) \subseteq \mathcal{S}_{\rho_2, \delta_2}^{m_2}(\varphi, T^*\mathcal{M} \times \mathcal{M}). \quad (3.11)$$

One observes immediately that $\bigcap_{m \in \mathbb{R}} \mathcal{S}_{1,0}^m(\varphi, T^*\mathcal{M} \times \mathcal{M})$ is independent of the choice φ . Therefore, we denote

$$\mathcal{S}^{-\infty}(T^*\mathcal{M} \times \mathcal{M}) = \bigcap_{m \in \mathbb{R}} \mathcal{S}_{1,0}^m(\varphi, T^*\mathcal{M} \times \mathcal{M}) \quad \text{and}$$

$$\mathcal{S}^{-\infty}(T^*\mathcal{M}) = \bigcap_{m \in \mathbb{R}} \mathcal{S}_{1,0}^m(\varphi, T^*\mathcal{M}).$$

Example 3.1. A function $f(x, \xi; y) \in \mathcal{C}^\infty(T^*\mathcal{M} \times \mathcal{M})$ positively homogeneous of degree k in ξ is a φ -amplitude from the class $\mathcal{S}_{1,0}^k(\varphi, T^*\mathcal{M} \times \mathcal{M})$ for any global phase function φ .

A natural question concerning the classes $\mathcal{S}_{\rho,\delta}^m(\varphi, T^*\mathcal{M})$ and $\mathcal{S}_{\rho,\delta}^m(\varphi, T^*\mathcal{M} \times \mathcal{M})$ is the nature of their relationship with the classes $\mathcal{S}_{\rho,\delta}^m(T^*\mathcal{M} \times \mathcal{M})$ and $\mathcal{S}_{\rho,\delta}^m(T^*\mathcal{M} \times \mathcal{M})$ respectively, mentioned in Section 2.1. In fact, Example 3.1 seems to suggest that the definition of the classes of φ -amplitudes does not always depend on the choice of the phase function φ . We will prove that the definition of classes $\mathcal{S}_{\rho,\delta}^m(\varphi, T^*\mathcal{M})$ and $\mathcal{S}_{\rho,\delta}^m(\varphi, T^*\mathcal{M} \times \mathcal{M})$ does not depend on the choice of global phase functions under the classical assumption $1 - \rho \leq \delta$. Instead, when $1 - \rho > \delta$, Definition 3.5 defines a new class of symbols and amplitudes.

This is equivalent to noticing that, when $1 - \rho \leq \delta$, $\partial_\xi^\alpha \mathfrak{D}_x^\beta$ and $\partial_\xi^\alpha \partial_x^\beta$ produce the same estimate of the order of a φ -symbol. Before proving this result we need to introduce two technical lemmas.

Lemma 3.7. Let $a \in \mathcal{C}^\infty(T^*\mathcal{M})$, then for all multi-indices α and β

$$\partial_\xi^\alpha \mathfrak{D}_{\varphi,x}^\beta a(x, \xi) = \partial_\xi^\alpha \partial_x^\beta a(x, \xi) + \sum_{\substack{\nu \leq \alpha \\ \mu < \beta \\ |\mu| + |\mu'| \leq |\beta|}} p_{\alpha,\nu,\beta,\mu,\mu'}(x, \xi) \partial_\xi^{\nu+\mu'} \partial_x^\mu a(x, \xi), \quad (3.12)$$

with $p_{\alpha,\nu,\beta,\mu,\mu'}$ positively homogeneous of degree $|\mu'| + |\nu| - |\alpha|$ in ξ .

Proof. By definition

$$\mathfrak{D}_{\varphi,x}^\beta a(x, \xi) = \frac{d^\beta}{dy^\beta} a(y, -\nabla_y \varphi(x, \xi; y)) \Big|_{y=x}.$$

If $|\beta| = 1$, differentiation with respect to y might go either to the first or to the second variable of a . In this latter case, by the chain rule, we obtain that differentiation in y produces differentiation in ξ of a multiplied by a factor positively homogeneous of degree 1 in ξ because $-\nabla_y \varphi(x, \xi; y)$ is positively homogeneous of degree 1 in ξ . The case of higher order derivatives proceeds similarly. The only difference is the presence of a positively homogeneous factor in ξ . Notice that differentiation in y of such a factor does not change its degree of homogeneity.

Therefore, since $-\nabla_y \varphi(x, \xi; y) \Big|_{y=x} = \xi$, we obtain

$$\mathfrak{D}_{\varphi,x}^\beta a(x, \xi) = \partial_x^\beta a(x, \xi) + \sum_{\substack{\mu < \beta \\ |\mu| + |\mu'| \leq |\beta|}} \tilde{p}_{\beta,\mu,\mu'}(x, \xi) \partial_\xi^{\mu'} \partial_x^\mu a(x, \xi), \quad (3.13)$$

with $\tilde{p}_{\beta,\mu,\mu'}$ positively homogeneous of degree $|\mu'|$ in ξ .

In order to obtain (3.12), we just need to apply the Leibniz rule for differentiation to the sum in (3.13) and notice that each differentiation in ξ of $\tilde{p}_{\beta,\mu,\mu'}$ decreases its degree of homogeneity by 1. \square

Lemma 3.8. *Let $a \in \mathcal{C}^\infty(T^*\mathcal{M})$, then*

$$\partial_\xi^\alpha \partial_x^\beta a(x, \xi) = \partial_\xi^\alpha \mathfrak{D}_{\varphi,x}^\beta a(x, \xi) + \sum_{\substack{\nu \leq \alpha \\ \mu < \beta \\ |\mu| + |\mu'| \leq |\beta|}} q_{\alpha,\nu,\beta,\mu,\mu'}(x, \xi) \partial_\xi^{\nu+\mu'} \mathfrak{D}_{\varphi,x}^\mu a(x, \xi), \quad (3.14)$$

with $q_{\alpha,\nu,\beta,\mu,\mu'}$ positively homogeneous of degree $|\mu'| + |\nu| - |\alpha|$ in ξ .

Proof. We claim that

$$\partial_x^\beta a(x, \xi) = \mathfrak{D}_{\varphi,x}^\beta a(x, \xi) + \sum_{\substack{\mu < \beta \\ |\mu| + |\mu'| \leq |\beta|}} \tilde{q}_{\beta,\mu,\mu'}(x, \xi) \partial_\xi^{\mu'} \mathfrak{D}_{\varphi,x}^\mu a(x, \xi), \quad (3.15)$$

We prove this by induction on $|\beta|$.

The claim for $|\beta| = 1$ is a consequence of (3.8). Next, we need to notice that from (3.13) we have

$$\partial_x^\beta a(x, \xi) = \mathfrak{D}_{\varphi, x}^\beta a(x, \xi) - \sum_{\substack{\mu < \beta \\ |\mu| + |\mu'| \leq |\beta|}} \tilde{q}_{\beta, \mu, \mu'}(x, \xi) \partial_\xi^{\mu'} \partial_x^\mu a(x, \xi),$$

and by the induction hypothesis, since $\mu < \beta$ we can replace ∂_x^μ with operators which are linear combinations of $\partial_\xi^{\tilde{\mu}'} \mathfrak{D}_{\varphi, x}^{\tilde{\mu}} a$, with $\tilde{\mu} \leq \mu$ and $|\tilde{\mu}| + |\tilde{\mu}'| \leq |\mu|$, with coefficients positively homogeneous of degree $\tilde{\mu}'$ in ξ .

In order to obtain (3.14), we just need to apply the Leibniz rule for differentiation to the sum in (3.15) and notice that each differentiation in ξ of $\tilde{q}_{\beta, \mu, \mu'}$ decreases the degree of homogeneity by 1. \square

We now possess the necessary tools to state and prove the desired result.

Proposition 3.9. *Let $1 - \rho \leq \delta$ and $a \in \mathcal{C}^\infty(T^*\mathcal{M})$. Then, in any coordinate system $\{x^k\}$ for all multi-indices α and β there exists $C = C_{\mathcal{M}, \alpha, \beta} > 0$ such that*

$$|\partial_\xi^\alpha \mathfrak{D}_x^\beta a(x, \xi)| \leq C \langle \xi \rangle_x^{m - \rho|\alpha| + \delta|\beta|} \quad (3.16)$$

if and only if there exists $\tilde{C} = \tilde{C}_{\mathcal{M}, \alpha, \beta} > 0$ such that

$$|\partial_\xi^\alpha \partial_x^\beta a(x, \xi)| \leq \tilde{C} \langle \xi \rangle_x^{m - \rho|\alpha| + \delta|\beta|}. \quad (3.17)$$

Remark 3.10. If $1 - \rho \leq \delta$, then the definition of the classes $\mathcal{S}_{\rho, \delta}^m(\varphi, T^*\mathcal{M})$ and $\mathcal{S}_{\rho, \delta}^m(\varphi, T^*\mathcal{M} \times \mathcal{M})$ are independent of the choice of the global phase function φ and therefore we denote these classes simply by $\mathcal{S}_{\rho, \delta}^m(T^*\mathcal{M})$ and $\mathcal{S}_{\rho, \delta}^m(T^*\mathcal{M} \times \mathcal{M})$.

Proof (Proposition 3.9). Assume (3.16) holds true. By (3.14), we obtain immediately that in any coordinate system $\{x^k\}$

$$|\partial_\xi^\alpha \partial_x^\beta a(x, \xi)| \leq C \sum_{j=1}^{|\beta|} \langle \xi \rangle_x^{m-\rho|\alpha|+\delta j+(1-\rho)(|\beta|-j)}.$$

Since by assumption $1 - \rho \leq \delta$, inequality (3.17) follows immediately.

The other implication is proved analogously using (3.12). \square

In local coordinates, if $a \in S_{\rho,\delta}^m(X \times \mathbb{R}^n)$ then for any multi-indices α and β we know that $\partial_\theta^\alpha \partial_x^\beta a(x, \theta) \in S_{\rho,\delta}^{m-\rho|\alpha|+\delta|\beta|}(X \times \mathbb{R}^n)$. This property is fundamental in a number of applications. Therefore, it is desirable to have a similar property in the case of φ -symbols. The remaining part of this section is devoted to proving that if $a \in \mathcal{S}_{\rho,\delta}^m(\varphi, T^*\mathcal{M})$ and $1 - \rho \leq 2\delta$, then $\partial_\xi^\alpha \mathfrak{D}_{\varphi,x}^\beta a(x, \xi) \in \mathcal{S}_{\rho,\delta}^{m-\rho|\alpha|+\delta|\beta|}(\varphi, T^*\mathcal{M})$. We split this problem into two parts.

First, we focus on proving that $\partial_\xi^\alpha a(x, \xi) \in \mathcal{S}_{\rho,\delta}^{m-\rho|\alpha|}(\varphi, T^*\mathcal{M})$. This proof is rather simple and does not actually require the condition $1 - \rho \leq 2\delta$. Second, we show that $\mathfrak{D}_{\varphi,x}^\beta a(x, \xi) \in \mathcal{S}_{\rho,\delta}^{m+\delta|\beta|}(\varphi, T^*\mathcal{M})$. This result requires a little extra care because of the operator $\mathfrak{D}_{\varphi,x}^\beta$ and because the condition $1 - \rho \leq 2\delta$ originates here.

We begin by analysing the action of differentiation on the phase variable of a φ -symbol. The key result is the following.

Proposition 3.11. *Let $a(x, \xi) \in \mathcal{S}_{\rho,\delta}^m(\varphi, T^*\mathcal{M})$. Then, for all multi-indices α' we have*

$$\partial_\xi^{\alpha'} a \in \mathcal{S}_{\rho,\delta}^{m-\rho|\alpha'|}(\varphi, T^*\mathcal{M}).$$

The proof of this proposition relies on the following lemma. Roughly speaking, we prove that the order in which we apply the differential operators ∂_ξ^α and $\mathfrak{D}_{\varphi,x}^\beta$ to a φ -symbol does not change its estimate.

Lemma 3.12. *Let $a \in \mathcal{C}^\infty(T^*\mathcal{M})$. In any coordinate system $\{x^k\}$ there exist $C_{\mathcal{M},\alpha,\beta} > 0$ such that*

$$|\partial_\xi^\alpha \mathfrak{D}_{\varphi,x}^\beta a(x, \xi)| \leq C_{\mathcal{M},\alpha,\beta} \langle \xi \rangle_x^{m-\rho|\alpha|+\delta|\beta|}, \quad (3.18)$$

for all multi-indices α and β , if and only if there exist $\tilde{C}_{\mathcal{M},\alpha,\beta} > 0$ such that

$$|\mathfrak{D}_{\varphi,x}^\beta \partial_\xi^\alpha a(x, \xi)| \leq \tilde{C}_{\mathcal{M},\alpha,\beta} \langle \xi \rangle_x^{m-\rho|\alpha|+\delta|\beta|}, \quad (3.19)$$

for all multi-indices α and β .

Proof. Note that for all nonzero multi-indices α

$$\frac{d^\alpha}{d\xi^\alpha} a(y, -\nabla_y \varphi(x, \xi; y)) = \sum_{1 \leq |\nu| \leq |\alpha|} p_{\alpha,\nu}(x, \xi; y) (\partial_\xi^\nu a)(y, -\nabla_y \varphi(x, \xi; y)), \quad (3.20)$$

with $p_{\alpha,\nu}$ positively homogeneous of degree $|\nu| - |\alpha|$ in ξ . Indeed, by the chain rule, the first differentiation produces derivatives of a in the second variable multiplied by factors positively homogeneous of degree 0 in ξ (because $\partial_{\xi_j} \partial_{y_k} \varphi(x, \xi; y)$ is positively homogeneous of degree 0 in ξ). The case of higher order derivatives proceeds similarly. The only difference is the presence of positively homogeneous coefficients, whose degree of homogeneity is decreased by 1 after each differentiation in ξ .

By Definition 3.4, by switching the order of differentiation we obtain

$$\partial_\xi^\alpha \mathfrak{D}_{\varphi,x}^\beta a(x, \xi) = \frac{d^\beta}{dy^\beta} \frac{d^\alpha}{d\xi^\alpha} a(y, -\nabla_y \varphi(x, \xi; y)) \Big|_{y=x}.$$

Applying (3.20) and using the Leibniz rule, we obtain

$$\partial_\xi^\alpha \mathfrak{D}_{\varphi,x}^\beta a(x, \xi) = \mathfrak{D}_{\varphi,x}^\beta \partial_\xi^\alpha a(x, \xi) + \sum_{\substack{\mu < \beta \\ 1 \leq |\nu| \leq |\alpha|}} p_{\alpha,\nu,\beta,\mu}(x, \xi) \mathfrak{D}_{\varphi,x}^\mu \partial_\xi^\nu a(x, \xi), \quad (3.21)$$

with $p_{\alpha,\nu,\beta,\mu}$ positively homogeneous of degree $|\nu| - |\alpha|$ in ξ .

From (3.21), it follows immediately that (3.19) implies (3.18).

The other implication can be proved easily by induction over $|\beta|$ since from (3.21) it follows that

$$\mathfrak{D}_{\varphi,x}^\beta \partial_\xi^\alpha a(x, \xi) = \partial_\xi^\alpha \mathfrak{D}_{\varphi,x}^\beta a(x, \xi) - \sum_{\substack{\mu < \beta \\ 1 \leq |\nu| \leq |\alpha|}} p_{\alpha,\nu,\beta,\mu}(x, \xi) \mathfrak{D}_{\varphi,x}^\mu \partial_\xi^\nu a(x, \xi).$$

We only need to notice that on the right hand side $|\mu|$ is at most $|\beta| - 1$ in the operator $\mathfrak{D}_{\varphi,x}^\mu \partial_\xi^\nu$. \square

Proof (Proposition 3.11). It is enough to notice that by Lemma 3.12 for all multi-indices α and β there exist $C_{\mathcal{M},\alpha,\alpha',\beta} > 0$ such that

$$|\partial_\xi^\alpha \mathfrak{D}_{\varphi,x}^\beta \partial_\xi^{\alpha'} a(x, \xi)| \leq C_{\mathcal{M},\alpha,\alpha',\beta} \langle \xi \rangle_x^{m-\rho(|\alpha|+|\alpha'|)+\delta|\beta|}$$

if and only if there exist $\tilde{C}_{\mathcal{M},\alpha,\alpha',\beta} > 0$ such that

$$|\partial_\xi^{\alpha+\alpha'} \mathfrak{D}_{\varphi,x}^\beta a(x, \xi)| \leq \tilde{C}_{\mathcal{M},\alpha,\alpha',\beta} \langle \xi \rangle_x^{m-\rho(|\alpha|+|\alpha'|)+\delta|\beta|}.$$

\square

Now, we turn our attention to the action of the operator $\mathfrak{D}_{\varphi,x}^\beta$ over a φ -symbol. The goal is to prove the following result.

Proposition 3.13. *Let $a(x, \xi) \in \mathcal{S}_{\rho,\delta}^m(\varphi, T^*\mathcal{M})$. If $1 - \rho \leq 2\delta$, then for all multi-indices β'*

$$\mathfrak{D}_{\varphi,x}^{\beta'} a \in \mathcal{S}_{\rho,\delta}^{m+\delta|\beta'|}(\varphi, T^*\mathcal{M}). \quad (3.22)$$

The condition $1 - \rho \leq 2\delta$ constitutes an improvement with respect to the classical condition for symbolic calculus on manifolds, $1 - \rho \leq \delta$. In order to prove Proposition 3.13, we need the following technical lemma.

Lemma 3.14. *Let $a \in \mathcal{C}^\infty(T^*\mathcal{M})$, then*

$$\begin{aligned} & \nabla_y a(y, -\nabla_y \varphi(x, \xi; y)) \\ &= (\nabla_{\varphi,x} a)(y, -\nabla_y \varphi(x, \xi; y)) + M_\varphi(x, \xi; y)(\nabla_\xi a)(y, -\nabla_y \varphi(x, \xi; y)), \end{aligned}$$

where $M_\varphi = [m_{k,l}(x, \xi; y)]_{k,l=1}^n$ is a $n \times n$ matrix with positively homogeneous entries of degree 1 in ξ and

$$M_\varphi(x, \xi; y) \Big|_{y=x} = 0. \quad (3.23)$$

Proof. This is an immediate consequence of (3.8) and the properties of global phase functions. We observe that

$$M_\varphi(x, \xi; y) = H_y \varphi(x, \xi; y) - H_z \varphi(y, -\nabla_y \varphi(x, \xi; y); z) \Big|_{z=y}.$$

□

Proof (Proposition 3.13). The proof proceeds by induction over $|\beta'|$.

Let $|\beta'| = 1$, e.g. $\beta' = e_k$. By Lemma 3.14, it follows that for any multi-index β

$$\begin{aligned} \mathfrak{D}_{\varphi,x}^{\beta+e_k} a(x, \xi) &= \frac{d^\beta}{dy^\beta} \frac{d}{dy_k} a(y, -\nabla_y \varphi(x, \xi; y)) \Big|_{y=x} \\ &= \mathfrak{D}_{\varphi,x}^\beta \mathfrak{D}_{\varphi,x}^{e_k} a(x, \xi) \\ &\quad + \sum_{j=1}^n \frac{d^\beta}{dy^\beta} \left[m_{k,j}(x, \xi; y) (\partial_{\xi_j} a)(y, -\nabla_y \varphi(x, \xi; y)) \right]_{y=x}. \end{aligned}$$

By Lemma 3.14, $m_{k,j}(x, \xi; x) = 0$. Therefore, at least one derivative in y must go to $m_{k,j}$ in order for the corresponding term in the sum not to be zero. Thus, we have

$$\mathfrak{D}_{\varphi,x}^\beta \mathfrak{D}_{\varphi,x}^{e_k} a(x, \xi) = \mathfrak{D}_{\varphi,x}^{\beta+e_k} a(x, \xi) - \sum_{\substack{\mu < \beta \\ |\nu| = 1}} p_{\varphi,\nu,\beta,\mu}(x, \xi) \mathfrak{D}_{\varphi,x}^\mu \partial_\xi^\nu a(x, \xi), \quad (3.24)$$

with $p_{\varphi,\nu,\beta,\mu}(x, \xi)$ positively homogeneous of degree 1 in ξ . The equality (3.24) and Lemma 3.12 imply that $\mathfrak{D}_{\varphi,x}^{e_k} a(x, \xi) \in \mathcal{S}_{\rho,\delta}^{m+\delta}(\varphi, T^* \mathcal{M})$ provided $1 - \rho \leq 2\delta$.

For the case of general β' , it is sufficient to use (3.24) and Proposition 3.11, together with the inductive assumption that $\mathfrak{D}_{\varphi,x}^\nu a(x, \xi) \in \mathcal{S}_{\rho,\delta}^{m+\delta|\nu|}(\varphi, T^* \mathcal{M})$ for all $|\nu| < |\beta'|$. \square

We conclude this section by noticing that combining Proposition 3.11 and Proposition 3.13 one obtains that for all multi-indices α and β

$$\partial_\xi^\alpha \mathfrak{D}_{\varphi,x}^\beta a(x, \xi) \in \mathcal{S}_{\rho,\delta}^{m-\rho|\alpha|+\delta|\beta|}(\varphi, T^* \mathcal{M}),$$

when $a \in \mathcal{S}_{\rho,\delta}^m(\varphi, T^* \mathcal{M})$ and $1 - \rho \leq 2\delta$.

3.3 Pseudodifferential Operators

In this section, we use φ -symbols and phase functions introduced previously to propose an alternative method to define pseudodifferential operators on manifolds. In order to provide an intrinsic integral representation, we deal with operators acting in the space of densities, according to the approach already proposed by Safarov in [16].

Roughly speaking, a density is a spatially varying quantity on the manifold, that behaves nicely under change of coordinates.

Definition 3.15. A \varkappa -density u on \mathcal{M} , $\varkappa \in \mathbb{R}$, is a section of a line bundle on \mathcal{M} which under change of coordinates on \mathcal{M} behaves according to the following rule:

$$u(y) = |\det\{dx^j/dy^k\}|^\varkappa u(x(y)).$$

One can observe that the usual functions on \mathcal{M} are 0-densities. We denote by $\mathcal{C}^\infty(\mathcal{M}, \Omega^\varkappa)$ and $\mathcal{C}_0^\infty(\mathcal{M}, \Omega^\varkappa)$, the space of smooth \varkappa -densities and smooth \varkappa -densities with compact support respectively. In case $\varkappa = 0$, we simply write $\mathcal{C}^\infty(\mathcal{M})$ and $\mathcal{C}_0^\infty(\mathcal{M})$ respectively.

If $u \in \mathcal{C}_0^\infty(\mathcal{M}, \Omega^\varkappa)$ and $v \in \mathcal{C}^\infty(\mathcal{M}, \Omega^{1-\varkappa})$, then the product uv is a 1-density and the integral $\int_{\mathcal{M}} uv \, dx$ has an intrinsic meaning because its value is independent of the choice of local coordinates. This allows us to define the inner product

$$\langle u, v \rangle_{L^2} = \int_{\mathcal{M}} u \bar{v} \, dx \tag{3.25}$$

on the space of $1/2$ -densities and to introduce the Hilbert space $L^2(\mathcal{M}, \Omega^{1/2})$ in the standard way.

Let $A : \mathcal{C}_0^\infty(\mathcal{M}, \Omega^{1/2}) \rightarrow \mathcal{C}^\infty(\mathcal{M}, \Omega^{1/2})$ be a linear operator with the Schwartz kernel $\mathcal{A}(x, y)$. Traditionally, the operator A is said pseudodifferential if

- $\mathcal{A}(x, y)$ is smooth outside the diagonal in $\mathcal{M} \times \mathcal{M}$;
- in each coordinate patch $U \times U \subset \mathcal{M} \times \mathcal{M}$ where $\{y^k\}$ are the same coordinates as $\{x^k\}$, if the kernel $\mathcal{A}(x, y)$ is represented modulo a smooth function by an oscillatory integral of the form

$$\int_{\mathbb{R}^n} e^{i\langle x-y, \theta \rangle} a(x, y, \theta) d\theta, \quad (3.26)$$

where $a(x, \theta, y)$ is an amplitude from some coordinate class $S_{\rho, \delta}^m(U \times U \times \mathbb{R}^n)$, with $\rho, \delta \in [0, 1]$ (without assuming $\rho > \delta$).

We associate with a φ -amplitude $a \in \mathcal{S}_{\rho, \delta}^m(\varphi, T^*\mathcal{M} \times \mathcal{M})$ the oscillatory integral

$$\mathcal{A}_\varphi(x, y) = \mathfrak{p}(x; y) \int_{T_x^*\mathcal{M}} e^{i\varphi(x, \xi; y)} a(x, \xi) d\xi, \quad (3.27)$$

where \mathfrak{p} is a weight factor such that $\mathfrak{p}(x, \cdot)$ is in $\mathcal{C}^\infty(\mathcal{M}, \Omega^{1/2})$ and $\mathfrak{p}(\cdot, y)$ is in $\mathcal{C}^\infty(\mathcal{M}, \Omega^{-1/2})$.

One notices immediately that for any $u \in \mathcal{C}^\infty(\mathcal{M}, \Omega^{1/2})$

$$Au(x) = \int e^{i\varphi(x, \xi; y)} \mathfrak{p}(x; y) a(x, \xi) u(y) dy d\xi \in \mathcal{C}^\infty(\mathcal{M}, \Omega^{1/2}).$$

and integration does not depend on the choice of local coordinates. In addition, one observes that the symbol a depends on the choice of the weight factor \mathfrak{p} . This dependence will not be reflected in the notation, but it will be always implicitly assumed.

More generally one can consider operators acting in the space of \varkappa -densities. In this case (see for example [16]), we need $\mathbf{p}(x; y)$ to be a $(\varkappa - 1)$ -density in x and a $(1 - \varkappa)$ -density in y . We emphasise the fact that we have great freedom in the choice of factor \mathbf{p} . For our future purposes, it is convenient to assume

$$\mathbf{p}(x, x) \equiv 1.$$

Notice that \mathbf{p} behaves as a function on the diagonal in $\mathcal{M} \times \mathcal{M}$ and hence this is a legitimate choice.

By (3.5), in a coordinate patch $U \times U \subseteq \mathcal{M} \times \mathcal{M}$ where $\{y^k\}$ is the same coordinates as $\{x^k\}$

$$\varphi(x, \xi; y) = \langle x - y, \xi \rangle + \langle x - y, V(x, \xi; y) \rangle,$$

for some vector-valued function V on $U \times \mathbb{R}^n \setminus \{0\} \times U$ positively homogeneous of degree 1 in ξ and vanishing on the diagonal $\mathcal{D} = \{(x, y) \in U \times U : x = y\}$. Let

$$\tilde{\xi} = \tilde{\xi}(x, \xi; y) = \xi + V(x, \xi; y).$$

Notice that on the diagonal \mathcal{D} one has $\det\{d\tilde{\xi}^j/d\xi^k\} = 1$. Therefore, for x and y sufficiently close $\tilde{\xi}$ has an inverse as a function of ξ . Denote its inverse by $\xi = \xi(x, \tilde{\xi}; y)$, then

$$\int e^{i\varphi(x, \xi; y)} a(x, \xi) d\xi = \int e^{i\langle x - y, \tilde{\xi} \rangle} a(x, \xi(x, \tilde{\xi}; y)) |\det\{d\xi^j/d\tilde{\xi}^k\}| d\tilde{\xi}. \quad (3.28)$$

We observe that

$$a(x, \xi(x, \tilde{\xi}; y)) |\det\{d\xi^j/d\tilde{\xi}^k\}| \in S_{\rho, \delta'}^m(U \times \mathbb{R}^n \times U),$$

where $\delta' = \max\{\delta, 1 - \rho\}$. In fact, $|\det\{d\xi^j/d\tilde{\xi}^k\}|$ and $\xi(x, \tilde{\xi}; y)$ are homogeneous of degree 0 and 1 respectively in $\tilde{\xi}$. Therefore $\mathcal{A}_\varphi(x, y)$ coincides in a small neighbourhood of the diagonal in $\mathcal{M} \times \mathcal{M}$ with the Schwartz kernel of a Ψ DO. This Ψ DO acts in the space of $1/2$ -densities on \mathcal{M} and is determined uniquely modulo an operator with smooth kernel.

Definition 3.16. We denote by $L_{\rho, \delta}^m(\varphi, \mathcal{M}, \Omega^{1/2})$ the class of Ψ DOs A acting in the space of $1/2$ -densities on \mathcal{M} whose Schwartz kernel is represented in a neighbourhood of the diagonal by an oscillatory integral of the form (3.27). We denote by $L^{-\infty}(\mathcal{M}, \Omega^{1/2})$ the class of operators with smooth kernels acting in the space of $1/2$ -densities generated by $\mathcal{S}^{-\infty}(T^*\mathcal{M})$.

In local coordinates the oscillatory integral (3.27) admits the standard regularisation (see for example [11]).

We can replace the symbol in (3.27) by an amplitude $a \in \mathcal{S}_{\rho, \delta}^m(\varphi, T^*\mathcal{M} \times \mathcal{M})$. Obviously also in this case the oscillatory integral coincides in a neighbourhood of the diagonal in $\mathcal{M} \times \mathcal{M}$ with a Ψ DO. Similar properties to the ones presented in Proposition 3.6 can be easily proved for a Ψ DO in $L_{\rho, \delta}^m(\varphi, \mathcal{M}, \Omega^{1/2})$.

If $1 - \rho \leq \delta$ then, by [11, Theorem 19.1] the classes $L_{\rho, \delta}^m(\varphi, \mathcal{M}, \Omega^{1/2})$ are independent of φ and coincide with the standard classes of Ψ DOs. For, $1 - \rho > \delta$, generally speaking, the classes of Ψ DOs corresponding to different global phase functions φ are different. If the amplitude a is from $\mathcal{S}^{-\infty}(T^*\mathcal{M} \times \mathcal{M})$ then the oscillatory integral (3.27) determines a smooth density and the corresponding Ψ DO belongs to $L^{-\infty}(\mathcal{M}, \Omega^{1/2})$.

Furthermore, let $L^m(\mathcal{M}, \Omega^{1/2})$ be the class of Ψ DOs acting in the space of $1/2$ -densities such that the amplitudes in the corresponding local oscillatory integrals belong to $\mathcal{S}^m(T^*\mathcal{M})$. These operators are called classical Ψ DOs.

Finally, given $A \in L_{\rho,\delta}^m(\varphi, \mathcal{M}, \Omega^{1/2})$ we denote by A^* its formal adjoint with respect to the inner product defined in (3.25). Notice that if the kernel of A is the same as in (3.27), then the kernel of A^* is formally described by the oscillatory integral

$$\mathcal{A}^*(x, y) = \overline{\mathbf{p}(y; x)} \int e^{-i\varphi(y, \eta; x)} \overline{a(y, \eta)} d\eta. \quad (3.29)$$

At first glance one can immediately observe that this kernel relies on a representation based on a ‘dual symbol’. Moreover, it is not possible to turn (3.29) into an oscillatory integral of the form (3.27) by means of a change of variables and conclude straight away that A^* is a pseudodifferential operator in $L_{\rho,\delta}^m(\varphi, \mathcal{M}, \Omega^{1/2})$. We postpone the proof of this important fact to a more later point in our discussion.

Chapter 4

Stationary Phase Formula

In this chapter, we embark on the study of distributions of the kind

$$\begin{aligned} \int e^{i\tau\psi(x;y)} e^{-i\varphi(x,\xi;y)} p(x;y) \sigma(x,\xi) dy d\xi \\ = \tau^n \int e^{i\tau[\psi(x;y) - \varphi(x,\xi;y)]} p(x;y) \sigma(x,\tau\xi) dy d\xi \quad \tau > 0, \end{aligned} \quad (4.1)$$

where $\sigma \in \mathcal{S}_{\rho,\delta}^m(\varphi, T^*\mathcal{M})$ and $\psi, p \in C^\infty(\mathcal{M} \times \mathcal{M})$. This is central for our future purposes. In particular, the formulae concerning compositions and adjoints of Ψ DOs heavily depend on the results contained in this chapter.

This problem calls for the use of the method of stationary phase. Our analysis relies on the proof of this technique as outlined in [4, Section 7.7]. In fact, the argument outlined by Hörmander provides an effective approach for computing the quite complicated expressions of the terms constituting the expansion and describes in full detail the remainder term of the asymptotic expansion.

In the first section, we address a preliminary problem concerning the convergence of (4.1). This issue is only apparent because we can introduce a cut-off function in

the integral without affecting the asymptotic behaviour of the oscillatory integral. In addition, we prove that under suitable conditions on the phase function and the φ -amplitude in (4.1), the asymptotic behaviour for τ large is better than one expects. The proof relies on the same technique used to regularise an oscillatory integral in Section 1.1 based on integration by parts.

In the second section, we analyse the case where the phase function $\tau\psi(x; y) - \varphi(x, \xi; y)$ in (4.1) is quadratic or, equivalently, the case when the integrand is a Gaussian function. We derive an expansion of the oscillatory integral for this special case simply by using Fourier transform and Taylor's expansion. It consists again of a sum of terms obtained by applying a differential operator to the symbol σ plus a remainder term. Moreover, we prove that we can always express an oscillatory integral of the form (4.1) as a sum of Gaussian integrals modulo an error term. We make use of the result obtained in the first section to estimate the order of this term. We show that the condition $\rho > 1/3$ on the type of the amplitude guarantees that the expansion is asymptotic, that is, the order of the remainder decreases to $-\infty$ as the number of terms in the expansion increases.

Finally, combining the two results contained in the second section, we derive the formula for a general oscillatory integral of the kind (4.1). This is done in the last section of this chapter. Particular attention will be devoted to control the order of the remainder term originating from the expansion of a Gaussian integral.

From now on, we reserve the notation $H_x u(x)$ for the Hessian matrix of $u \in \mathcal{C}^2(\mathcal{M})$. Moreover, for a non-degenerate and symmetric $M \in \mathbb{R}^{n \times n}$ with eigenvalues λ_j , $j = 1, \dots, n$ define the signature of M as $\text{sgn } M = \sum_j \text{sgn } \lambda_j$. Throughout this chapter, we reserve the notation p for a function $p \in \mathcal{C}^\infty(\mathcal{M} \times \mathcal{M})$.

We shall assume that all the considerations are carried out in local coordinates. In fact, we can split the oscillatory integrals considered into the sum of integrals whose amplitudes have small supports. Therefore, without loss of generality, all the variables run over subsets of \mathbb{R}^n and densities are identified with functions. Finally, when results hold uniformly with respect to some parameters, the dependence on these parameters is not reflected in the notation. This will be often the case for the variable x .

4.1 Asymptotic Behaviour

We want to investigate the asymptotic behaviour when $\tau \rightarrow \infty$ of

$$\int e^{i\tau\psi(x;y)} e^{-i\varphi(x,\xi;y)} p(x;y) \sigma(x,\xi) dy d\xi, \quad (4.2)$$

where φ is a global phase function, as in Definition 3.2, $\psi \in \mathcal{C}^\infty(\mathcal{M} \times \mathcal{M})$ and $\sigma \in \mathcal{S}_{\rho,\delta}^m(\varphi, T^*\mathcal{M})$. Notice that

$$|\partial_\xi^\alpha \mathfrak{D}_{\varphi,x}^\beta \partial_y^\gamma (p(x,y) \sigma(x,\xi))| \leq C_{\varphi,\alpha,\beta,\gamma} \langle \xi \rangle_x^{m-\rho|\alpha|+\delta\beta}.$$

The method of stationary phase develops a technique which is useful for our purposes. In fact, the stationary phase formula provides an asymptotic expansion for integrals of the kind

$$\int e^{i\tau\Phi(z)} a(z) dz, \quad \tau > 0, \quad (4.3)$$

where Φ is a real \mathcal{C}^∞ function on \mathbb{R}^n , $a \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ and $\tau \rightarrow \infty$. If Φ has no critical points on the support of a , then (4.3) is rapidly decreasing as $\tau \rightarrow \infty$.

Therefore, the essential contributions for the study of the asymptotic behaviour must always come from points where the phase Φ is stationary. Generally, one makes the additional hypothesis that all the critical points of Φ are non-degenerate, i.e. if $\nabla_x \Phi(x_0) = 0$ at $x = x_0$, then the Hessian matrix of Φ is non-singular at x_0 . This implies that there are only finitely many isolated singularities. Therefore, by means of a partition of unity, one can decompose the amplitude a into a sum $a_1 + \dots + a_r$ and the support of each a_j contains only one critical point of Φ . We are thus reduced to analysing the case where Φ has only one critical point on the support of a .

We refer the interested reader to [6, Section 7.7] for a detailed account of this method. Instead we limit to recalling the following version of the stationary phase formula, which is useful for our purposes. This is [6, Theorem 7.7.5] where we used a notation consistent with our work.

Theorem 4.1. *Let $K \subset \mathbb{R}^n$ be a compact set and X an open neighbourhood of K . If $a \in \mathcal{C}_0^\infty(K)$, $\Phi \in \mathcal{C}^\infty(X)$ real valued such that $\nabla_z \Phi(z_0) = 0$, $\nabla_z \Phi(z) \neq 0$ in $K \setminus \{z_0\}$, $|\det H_z \Phi(z_0)| = 1$ and $\text{sgn } H_z \Phi(z_0) = 0$, then for any $N \in \mathbb{N}$*

$$\left| \int e^{i\tau\Phi(z)} a(z) dz - \left(\frac{2\pi}{\tau} \right)^{n/2} e^{i\tau\Phi(z_0)} \sum_{j=0}^{N-1} \tau^{-j} P_j a(z_0) \right| \leq C \langle \tau \rangle^{-N} \sum_{|\alpha| \leq 2N} \sup |\partial_z^\alpha a(z)|,$$

for $\tau > 0$. Here C is bounded when Φ stays in a bounded set of $\mathcal{C}^\infty(X)$ and $|z - z_0|/|\nabla_z \Phi(z)|$ has a uniform bound. We have

$$P_j a = \sum_{\nu - \mu = j} \sum_{2\nu \geq 3\mu} \frac{i^{-j} 2^{-\nu}}{\mu! \nu!} \langle -(H_z^{-1} \Phi(z_0) \nabla_z), \nabla_z \rangle^\nu (g_{z_0}^\mu a)(z_0),$$

where

$$g_{z_0}(z) = \Phi(z) - \Phi(z_0) - \frac{1}{2} \langle H_z \Phi(z_0)(z - z_0), z - z_0 \rangle, \quad (4.4)$$

which vanishes of third order at z_0 and P_j is a differential operator of order $2j$ acting on a .

Remark 4.2. We observe that

$$\left| \int e^{i\tau\Phi(z)} a(z) dz \right| \leq \int |a(z)| dz \leq M < \infty.$$

However, with the additional assumption a has a zero of order $2N$ at z_0 , Theorem 4.1 implies

$$\left| \int e^{i\tau\Phi(z)} a(z) dz \right| \leq C \langle \tau \rangle^{-N}.$$

We notice that the asymptotic behaviour of the oscillatory integral for τ large is better than one expects. Integration by parts of factors of the kind $\mathcal{O}(|z - z_0|)$ is once again the reason for this.

The proof of Theorem 4.1 consists of three main steps:

- reduction of (4.3) to sum of Gaussian integrals;
- Taylor's expansion of each Gaussian integral;
- manipulations of Gaussian integrals through the Fourier transform.

Using the same strategy, we aim to prove a similar result for the oscillatory integral of the kind (4.2).

Before starting to prove the main result, we make some considerations regarding convergence and asymptotic behaviour for $\tau \rightarrow \infty$ of (4.2).

First of all, we address the issue concerning convergence. As noted in Section 1.1, oscillatory integrals do not generally converge in the usual sense. On the contrary, integrals considered in Theorem 4.1 converge absolutely. However, this can be easily fixed by introducing in the integrand of (4.2) a suitable cut-off function, which does not change the asymptotic behaviour of the integral. What follows is a simple adjustment of the technique presented by Safarov and Vassiliev in [24, Appendix C] for oscillatory integrals of the same kind on \mathbb{R}^n with classical amplitudes.

Lemma 4.3. *Let $\psi(x; y) \in \mathcal{C}^\infty(\mathcal{M})$ such that $\nabla_y \psi(x; y) \neq 0$. Denote*

$$\Omega_{\varphi, \psi} = \{(x, \xi; y) \in T^*\mathcal{M} \times \mathcal{M} : |\nabla_y \psi(x; y)|/3 \leq |\nabla_y \varphi(x, \xi; y)| \leq 3|\nabla_y \psi(x; y)|\}$$

and let $C_{\mathcal{D}} = \{(x, \xi; y) \in T^*\mathcal{M} \times \mathcal{M} : x = y\}$. Let $\chi(x, \xi; y) \in \mathcal{C}^\infty(T^*\mathcal{M} \times \mathcal{M})$ be an arbitrary cut-off function bounded with all its derivatives such that $\chi(x, \xi; y) = 1$ in a neighbourhood of the set $\Omega_{\varphi, \psi} \cap C_{\mathcal{D}}$. Then, for all $N \in \mathbb{N}$ there exists a positive constant C_N such that

$$\left| \int e^{i\tau\psi(x; y)} e^{-i\varphi(x, \xi; y)} p(x; y) \sigma(x, \xi) dy d\xi - \int e^{i\tau\psi(x; y)} e^{-i\varphi(x, \xi; y)} p(x; y) \chi(x, \xi/\tau; y) \sigma(x, \xi) dy d\xi \right| \leq C_N \langle \tau \rangle^{-N}. \quad (4.5)$$

Proof. First, we observe that if $\sigma \in \mathcal{S}^{-\infty}(T^*\mathcal{M} \times \mathcal{M})$, then both integrals on (4.5) are rapidly decreasing. Define

$$\mathcal{I}(x, y) = p(x; y) \int e^{-i\varphi(x, \xi; y)} \sigma(x, \xi) d\xi,$$

Let $L(x; y, \partial_y) = -i|\nabla_y \psi(x; y)|^{-2} \langle \nabla_y \psi(x; y), \nabla_y \rangle$, then integrating by parts we obtain

$$\begin{aligned} \int e^{i\tau\psi(x;y)} \mathcal{I}(x; y) dy &= \tau^{-N} \int L^N(e^{i\tau\psi(x;y)}) \mathcal{I}(x; y) dy \\ &= \tau^{-N} \int e^{i\tau\psi(x;y)} (L^T)^N \mathcal{I}(x; y) dy \end{aligned}$$

and similarly for the second integral. Therefore we can assume that all the amplitudes $a(x, \xi; y) = p(x, y)\sigma(x, \xi)$, which we are dealing with, vanish in a neighbourhood of the zero section of $T^*\mathcal{M} \times \mathcal{M}$. We can also assume that $\text{supp } a$ is supported in an arbitrarily small conic neighbourhood of $C_{\mathcal{D}}$. Let us choose this neighbourhood so small that $\chi = 1$ in a neighbourhood of the set $\Omega_{\varphi, \psi} \cap \text{supp } a$.

Next consider the integral

$$\int e^{i\tau\psi(x;y)} e^{-i\varphi(x, \xi; y)} p(x; y) (1 - \chi(x, \xi; y)) \sigma(x, \xi) dy d\xi. \quad (4.6)$$

On the support of the function $(1 - \chi(x, \xi; y))\sigma(x, \xi)$ we have the estimates

$$\begin{aligned} |\tau \nabla_y \psi(x; y) - \nabla_y \varphi(x, \xi; y)| &\geq \tau ||\nabla_y \psi(x; y)| - |\nabla_y \varphi(x, \xi/\tau; y)|| \\ &\geq \frac{\tau}{2} (|\nabla_y \psi(x; y)| + |\nabla_y \varphi(x, \xi/\tau; y)|) \geq C(\tau + \langle \xi \rangle_x), \end{aligned}$$

for some positive constant C . In the last inequality, we used the fact that $|\nabla_y \varphi(x, \xi; y)| \geq \tilde{C} \langle \xi \rangle_x$ on $\text{supp } a$, for some positive constant \tilde{C} . Now, we replace $e^{i\tau\psi(x;y)} e^{-i\varphi(x, \xi; y)}$ in (4.6) by

$$i\tau^{-1} \left\langle \nabla_y e^{i\tau\psi(x;y) - i\varphi(x, \xi; y)}, \frac{\tau \nabla_y \psi(x; y) - \nabla_y \varphi(x, \xi; y)}{|\tau \nabla_y \psi(x; y) - \nabla_y \varphi(x, \xi; y)|^{-2}} \right\rangle$$

and then integrate by parts with respect to y . Repeating this procedure, we

can transform (4.6) into an oscillatory integral with the same phase function and an amplitude which is estimated by all its derivatives by $C_N(\tau + \langle \xi \rangle_x)^{-N}$ for an arbitrary positive integer N . Obviously any such integral defines a smooth function rapidly decreasing with all its derivatives as $\tau \rightarrow \infty$. \square

By Lemma 4.3 we may insert in (4.2) a cut-off function χ . We can choose $\text{supp } \chi$ to be separated from the zero section. Then we obtain the integral

$$\begin{aligned} \int e^{i\tau\psi(x;y)} e^{-i\varphi(x,\xi;y)} p(x;y) \chi(x, \xi/\tau; y) \sigma(x, \xi) dy d\xi \\ = \tau^n \int e^{i\tau[\psi(x;y) - \varphi(x,\xi;y)]} p(x;y) \chi(x, \xi; y) \sigma(x, \tau\xi) dy d\xi, \end{aligned}$$

over a compact set which has the same asymptotic behaviour as (4.2). In other words, the two integrals are equal modulo $\mathcal{O}(\tau^{-\infty})$. From now until the end of the chapter, a cut-off function χ of the kind described in Lemma 4.3 is always implicitly inserted in the oscillatory integral which we intend to study. Although this modifies the amplitude in (4.2), one can observe that the fact that $\partial_y^\gamma \chi$ is bounded for any multi-index γ is sufficient for our purposes.

We now turn our attention to discussing the second point relevant to our analysis concerning the asymptotic behaviour when $\tau \rightarrow \infty$ of (4.8). Let

$$\Phi(\xi; y) = \psi(x, y) - \varphi(x, \xi; y),$$

with ψ as in Lemma 4.3. Dependence on x is not reflected in the notation since it can be regarded as a parameter. As mentioned earlier, we assume that Φ has a unique non-degenerate stationary point at $(\xi; y) = (\xi_0; y_0)$. For the sake of brevity from now until the end of the chapter, we introduce the following notation. Define the $2n$ -dimensional variable

$$z = (\xi - \xi_0; y - y_0)$$

Furthermore, let $dz = dy d\xi$ and $\nabla_z = (\nabla_\xi, \nabla_y)$. Next let $\sigma \in \mathcal{S}_{\rho,\delta}^m(\varphi, T^*\mathcal{M})$, define

$$\Phi(z) = \psi(x; y) - \varphi(x, \xi; y), \quad a_\tau(z) = p(x; y)\sigma(x, \tau\xi). \quad (4.7)$$

Then the stationary point is $z = 0$ and integral (4.2) takes the form

$$\tau^n \int e^{i\tau\Phi(z)} a_\tau(z) dz. \quad (4.8)$$

A first element relevant in our investigation is the presence of the parameter τ in the amplitude of the oscillatory integral (4.8). This is in contrast with the classical method of stationary phase for integrals of the form (4.3), where the function a does not depend on τ . Further into our analysis, there exists a positive constant C such that $C^{-1}\langle\tau\rangle \leq \langle\tau\xi\rangle_x \leq C\langle\tau\rangle$ for all $\tau > 0$ over $\text{supp } \chi$, for χ as in Lemma 4.3. Therefore, for a_τ as in (4.7) we have the following estimates

$$|\partial_\xi^\alpha \partial_y^\beta a_\tau(z)| \leq C\langle\tau\rangle^{m+(1-\rho)|\alpha|}. \quad (4.9)$$

In the following lemma, we replicate the result presented in Remark 4.2 for the oscillatory integral (4.8).

Lemma 4.4. *Let $\sigma \in \mathcal{S}_{\rho,\delta}^m(\varphi, T^*\mathcal{M})$, $a_\tau(z) = p(x, y)\sigma(x, \tau\xi)$ and g is a smooth function with a zero of order N at $z = 0$. Then there exists a positive constant C , which does not depend on τ , such that*

$$\left| \int e^{i\tau\Phi(z)} g(z) a_\tau(z) dz \right| \leq C\langle\tau\rangle^{m-N\rho'}, \quad \tau > 0, \quad (4.10)$$

with $\rho' = \min\{1/2, \rho\}$.

Proof. Since $\nabla_z \Phi(0) = 0$ and $H_z \Phi(z)$ is non-degenerate at $z = 0$, by Taylor's expansion one has

$$\nabla_z \Phi(z) = \left[\int_0^1 (1-t) H_z \Phi(tz) dt \right] z.$$

We deduce that $z = M(z) \nabla_z \Phi(z)$ in a neighbourhood of $z = 0$, where $M(z)$ are some matrix functions. Therefore, there exist some smooth vector functions $\mathbf{c}^{(j)}(z)$, $j \in \{1 \dots, 2n\}$, such that

$$z_j = \langle \mathbf{c}^{(j)}(z), \nabla_z \Phi(z) \rangle, \quad (4.11)$$

where z_j is the j -th component of the $2n$ -dimensional vector z . If g has a zero order N at $z = 0$, again by Taylor's expansion one has

$$\int e^{i\tau \Phi(z)} g(z) a_\tau(z) dz = \sum_{|\gamma|=N} \int e^{i\tau \Phi(z)} z^\gamma g_\gamma(z) a_\tau(z) dz, \quad (4.12)$$

for some smooth functions g_γ .

Therefore, by (4.11) we rewrite each term $z^\gamma g_\gamma(z)$ in (4.12) in the form

$$\langle \mathbf{c}^{(1)}(z), \nabla_z \Phi(z) \rangle^{\gamma_1} \dots \langle \mathbf{c}^{(2n)}(z), \nabla_z \Phi(z) \rangle^{\gamma_{2n}} g_\gamma(z). \quad (4.13)$$

Now, pick one copy of $\langle \mathbf{c}^{(j)}(z), \nabla_z \Phi(z) \rangle$ and integrate by parts using

$$e^{i\tau \Phi(z)} \langle \nabla_z \Phi(z), \mathbf{c}^{(j)}(z) \rangle = -i\tau^{-1} \langle \nabla_z e^{i\tau \Phi(z)}, \mathbf{c}^{(j)}(z) \rangle.$$

Then one of the following occurs.

- We differentiate $\sigma(x, \tau \xi)$ with respect to ξ . This produces a factor τ , which cancels out with τ^{-1} obtained integrating by parts and reduce the order of

the amplitude σ by ρ . Furthermore, the new function has a zero of order $N - 1$ at $z = 0$ since we retain all the remaining copies of $\langle \mathbf{c}^{(j)}(z), \nabla_z \Phi(z) \rangle$.

- We differentiate another element of the form $\langle \mathbf{c}^{(k)}(z), \nabla_z \Phi(z) \rangle$, $k \in \{1 \dots, 2n\}$, with respect either ξ or y . The new function has a zero of order $N - 2$ at $z = 0$. We observe that we retain the factor τ^{-1} coming from integration by parts.
- We differentiate $\mathbf{c}^{(j)}$, p or g_γ with respect either ξ or y . We obtain a new function with a zero of order $N - 1$ at $z = 0$ and retain the factor τ^{-1} coming from integration by parts.

An analysis of the possibilities described above shows that, after all possible integrations by parts, we obtain a new amplitude for the integral in (4.12) of the form

$$\sum_{2k+j+l=N} \tau^{-k-j} q_{k,j,l}(x, \xi; y) \sigma_l(x, \tau \xi), \quad (4.14)$$

for some $\sigma_l \in \mathcal{S}_{\rho, \delta}^{m-\rho l}(\varphi, T^* \mathcal{M})$ and smooth functions $q_{k,j,l}$. Finally notice that (4.9) implies

$$|\tau^{-k-j} q_{k,j,l}(x, \xi; y) \sigma_l(x, \tau \xi)| \leq C \langle \tau \rangle^{m-k-j-\rho l} \leq C \langle \tau \rangle^{m-N\rho'}. \quad (4.15)$$

Combining (4.14) and (4.15), we obtain estimate (4.10). \square

4.2 Expansions of Gaussian Integrals

We recall that our final goal is to find the asymptotic behaviour for τ large of

$$\tau^n \int e^{i\tau \Phi(z)} a_\tau(z) dz. \quad (4.16)$$

Now we discuss the special case where Φ is a quadratic form, e.g. $\Phi(z) = \langle Qz, z \rangle / 2$ for some real symmetric non-singular matrix Q . In this case (4.16) is a Gaussian integral. Using a technique similar to the one used in the standard proof of the stationary phase formula, we are able to obtain an expansion for (4.16). All we need for finding the desired expression is Parseval's formula and the fact that the Fourier transform of Gaussian function is again a Gaussian function.

Lemma 4.5. *Let $\sigma \in \mathcal{S}_{\rho, \delta}^m(\varphi, T^*\mathcal{M})$ and $a_\tau(z) = p(x, y)\sigma(x, \tau\xi)$. Let $Q \in \mathbb{R}^{2n \times 2n}$ be a non-singular symmetric matrix such that $|\det Q| = 1$ and $\operatorname{sgn} Q = 0$. Let $P = -\langle Q^{-1}\nabla_z, \nabla_z \rangle$. Then*

$$\tau^n \int e^{\frac{1}{2}i\tau\langle Qz, z \rangle} a_\tau(z) dz = \sum_{j=0}^{M-1} \frac{(2i\tau)^{-j}}{j!} P^j a_\tau(0) + R^{(M)}(\tau),$$

where

$$R^{(M)}(\tau) = \frac{(2\pi)^{-n}(2i\tau)^{-M}}{(M-1)!} \int \int_0^1 e^{-\frac{1}{2}i\tau^{-1}t\langle Q^{-1}\tilde{z}, \tilde{z} \rangle} (1-t)^{M-1} \mathcal{F}_{z \rightarrow \tilde{z}}(P^M a_\tau(z)) dt d\tilde{z}.$$

Proof. Let $\mathcal{F}_{z \rightarrow \tilde{z}} = \mathcal{F}_{(\xi, y) \rightarrow (\tilde{\xi}, \tilde{y})}$ and $d\tilde{z} = d\tilde{y} d\tilde{\xi}$. Since $|\det Q| = 1$ and $\operatorname{sgn} Q = 0$, by [4, Theorem 7.6.1]

$$\begin{aligned} \mathcal{F}_{z \rightarrow \tilde{z}}(e^{-\frac{1}{2}i\tau\langle Qz, z \rangle}) &= (2\pi)^n \det(-i\tau^{-1}Q^{-1}) e^{\frac{1}{2}i\tau^{-1}\langle Q^{-1}\tilde{z}, \tilde{z} \rangle} \\ &= \left(\frac{2\pi}{\tau}\right)^n e^{\frac{1}{2}i\tau^{-1}\langle Q^{-1}\tilde{z}, \tilde{z} \rangle}. \end{aligned} \quad (4.17)$$

By Parseval's formula and (4.17), we have

$$\tau^n \int e^{\frac{1}{2}i\tau\langle Qz, z \rangle} a_\tau(z) dz = (2\pi)^{-2n} \int e^{-\frac{1}{2}i\tau^{-1}\langle Q^{-1}\tilde{z}, \tilde{z} \rangle} \mathcal{F}_{z \rightarrow \tilde{z}}(a_\tau(z)) d\tilde{z}. \quad (4.18)$$

Next, by Taylor's formula

$$e^{-\frac{1}{2}i\tau^{-1}\langle Q^{-1}\tilde{z}, \tilde{z} \rangle} = \sum_{j=0}^{M-1} \frac{(2i\tau)^{-j}}{j!} \langle Q^{-1}\tilde{z}, \tilde{z} \rangle^j + R_\tau^{(M)}(\tilde{z}), \quad (4.19)$$

where

$$R_\tau^{(M)}(\tilde{z}) = \frac{(2i\tau)^{-M}}{(M-1)!} \langle Q^{-1}\tilde{z}, \tilde{z} \rangle^M \int_0^1 e^{-\frac{1}{2}i\tau^{-1}t\langle Q^{-1}\tilde{z}, \tilde{z} \rangle} (1-t)^{M-1} dt.$$

Observe that

$$\langle Q^{-1}\tilde{z}, \tilde{z} \rangle^j \mathcal{F}_{z \rightarrow \tilde{z}}(a_\tau(z)) = \mathcal{F}_{z \rightarrow \tilde{z}} \left[\left(-\langle Q^{-1}\nabla_z, \nabla_z \rangle \right)^j a_\tau(z) \right]. \quad (4.20)$$

Substituting (4.19) into (4.18) and taking into account (4.20), we obtain

$$\begin{aligned} \tau^n \int e^{\frac{1}{2}i\tau\langle Qz, z \rangle} a_\tau(z) dz = & \sum_{j=0}^{M-1} \frac{(2i\tau)^{-j}}{j!} \int \mathcal{F}_{z \rightarrow \tilde{z}} \left[\left(-\langle Q^{-1}\nabla_z, \nabla_z \rangle \right)^j a_\tau(z) \right] d\tilde{z} \\ & + \int \Psi_\tau^M(\tilde{z}) \mathcal{F}_{z \rightarrow \tilde{z}} \left[\left(-\langle Q^{-1}\nabla_z, \nabla_z \rangle \right)^M a_\tau(z) \right] d\tilde{z}, \end{aligned}$$

with

$$\Psi_\tau^M(\tilde{z}) = \frac{(2\pi)^{-n}(2i\tau)^{-M}}{(M-1)!} \int_0^1 e^{-\frac{1}{2}i\tau^{-1}t\langle Q^{-1}\tilde{z}, \tilde{z} \rangle} (1-t)^{M-1} dt.$$

In order to conclude the proof, we just need to observe that for $g \in \mathcal{C}_0^\infty$, one has $g(0) = (2\pi)^{-n} \int \mathcal{F}_{z \rightarrow \tilde{z}}[g(z)] d\tilde{z}$. \square

Next, we show that a general oscillatory integral of the kind (4.16) can be expressed as the sum of Gaussian integrals plus a remainder term. Combining the following lemma with Lemma 4.5, we will derive the result in the general case.

Lemma 4.6. *Let $\sigma \in \mathcal{S}_{\rho,\delta}^m(\varphi, T^*\mathcal{M})$ and $a_\tau(z) = p(x, y)\sigma(x, \tau\xi)$. Assume Φ has a unique stationary point at $z = 0$ and $\det H_z\Phi(0) \neq 0$. Define $g_0(z) = \Phi(z) - \Phi(0) - \frac{1}{2}\langle H_z\Phi(0)z, z \rangle$. For all $N \in \mathbb{N}$,*

$$\int e^{i\tau\Phi(z)} a_\tau(z) dz = e^{i\tau\Phi(0)} \sum_{k=0}^{N-1} \frac{(i\tau)^k}{k!} \int e^{\frac{1}{2}i\tau\langle H_z\Phi(0)z, z \rangle} g_0^k(z) a_\tau(z) dz \mod \mathcal{O}(\tau^{m+N(1-3\rho')}), \quad (4.21)$$

with $\rho' = \min\{\rho, 1/2\}$.

Proof. Let $\Phi_s(z) = \Phi(0) + \frac{1}{2}\langle H_z\Phi(0)z, z \rangle + sg_0(z)$. Note that $\Phi_0(z) = \Phi(0) + \frac{1}{2}\langle H_z\Phi(0)z, z \rangle$ and $\Phi_1(z) = \Phi(z)$. Let us consider the integral

$$I_s(\tau) = \int e^{i\tau\Phi_s(z)} a_\tau(z) dz, \quad s \in [0, 1].$$

Clearly, $I_1(\tau)$ coincides with the left-hand side of (4.21) and $I_0(\tau)$ is a Gaussian integral. Furthermore,

$$I_s^{(k)}(\tau) = \frac{d^k}{ds^k} I_s(\tau) = (i\tau)^k \int e^{i\tau\Phi_s(z)} g_0^k(z) a_\tau(z) dz, \quad k \in \mathbb{N}.$$

Next, since $\nabla_z\Phi_s(0) = 0$ for all $s \in [0, 1]$, we observe that by Taylor's expansion one has

$$\nabla_z\Phi_s(z) = \left[\int_0^1 (1-t) H_z\Phi_s(tz) dt \right] z.$$

Therefore, we deduce that $z = M_s(z)\nabla_z\Phi_s(z)$ in a neighbourhood of $z = 0$, where $M_s(z)$ are some matrix functions bounded uniformly with respect to s with all their derivatives. Consequently, Lemma 4.4 is applicable to each phase function Φ_s and the estimates provided by the lemma are uniform with respect to s . This and

the fact that g_0^k has a zero of order $3k$ at $z = 0$ imply that

$$|I_s^{(k)}(\tau)| \leq C_k \langle \tau \rangle^{m+k(1-3\rho')}, \quad s \in [0, 1], \quad (4.22)$$

with C_k independent of s . Taylor's expansion of $I_s(\tau)$ at $s = 0$ is

$$I_s(\tau) = \sum_{k=0}^{N-1} \frac{1}{k!} I_s^{(k)}(0) s^k + R_s^{(N)}(\tau) s^{k+1}, \quad (4.23)$$

with

$$R_s^{(N)}(\tau) = \frac{1}{(N-1)!} \int_0^1 I_s^{(N)}(t\tau) (1-t)^{N-1} dt.$$

As a consequence of (4.22) we observe that

$$|R_s^{(N)}(\tau)| \leq \frac{1}{(N-1)!} \int_0^1 |I_s^{(N)}(t\tau)| (1-t)^{N-1} dt \leq C_N \langle \tau \rangle^{m+N(1-3\rho')},$$

with C_N independent of s . Therefore, (4.21) follows by setting $s = 1$ in (4.23). \square

4.3 Expansions of Oscillatory Integrals

As mentioned above, by combining Lemma 4.5 and Lemma 4.6 one can obtain an expansion for the integral (4.2). Since our final goal is to obtain an asymptotic expansion for this integral, we need to be able to obtain a sharp estimate for the remainder $R^{(M)}(\tau)$ defined in the previous section. Because of the importance of the information contained in the remainder terms for our future purposes, we derive a precise expression for this factor. We will make use of these expressions in the next chapter.

The following theorem is the central result of this chapter. It gives a prescription on how to obtain an expansion for (4.2). It can be regarded as a stationary phase formula for (4.2) and the similarity with Theorem 4.1 is evident.

Theorem 4.7. *Let Φ as in (4.7), $\sigma \in \mathcal{S}_{\rho,\delta}^m(\varphi, T^*\mathcal{M})$ and $a_\tau(z) = p(x, y)\sigma(x, \tau\xi)$. Assume Φ has a unique stationary point at $z = 0$ and $H_z\Phi(0) \neq 0$. Define*

$$g_0(z) = \Phi(z) - \Phi(0) - \frac{1}{2}\langle H_z\Phi(0)z, z \rangle,$$

which vanishes of third order at $z = 0$. Then, for all $N \in \mathbb{N}$

$$\begin{aligned} \tau^n \int e^{i\tau\Phi(z)} a_\tau(z) dz \\ = e^{i\tau\Phi(0)} \sum_{l=0}^{N-1} \sum_{j-k=l} \sum_{2j \geq 3k} \tau^{-l} \frac{i^{-l} 2^{-j}}{k!j!} P^j(g_0^k a_\tau)(0) + R^{(3N)}(\tau) \\ \text{mod } \mathcal{O}(\tau^{m+2N(1-3\rho')}), \end{aligned}$$

with $P = -\langle H_z^{-1}\Phi(0)z, z \rangle$ and

$$R^{(3N)}(\tau) = \sum_{k=0}^{2N-1} C_{N,k} \tau^{-3N+k} \int \Psi(\tilde{z}) \mathcal{F}_{z \rightarrow \tilde{z}}[P^{3N}(g_0^k a_\tau)(z)] d\tilde{z},$$

with $\Psi(\tilde{z}) = \int_0^1 e^{-\frac{1}{2}i\tau^{-1}t\langle -H_z^{-1}\Phi(0)\tilde{z}, \tilde{z} \rangle} (1-t)^{N-1} dt$.

Proof. Combining Lemma 4.6 and Lemma 4.5, we obtain

$$\begin{aligned} \tau^n \int e^{i\tau\Phi(z)} a_\tau(z) dz = \\ e^{i\tau\Phi(0)} \sum_{k=0}^{N-1} \frac{(i\tau)^k}{k!} \sum_{j=0}^{M-1} \frac{(2i\tau)^{-j}}{j!} P^j(g_0^k a_\tau)(0) + R^{(M)}(\tau) \\ \text{mod } \mathcal{O}(\tau^{m+N(1-3\rho')}). \quad (4.24) \end{aligned}$$

where

$$R^{(M)}(\tau) = \sum_{k=0}^{N-1} \frac{(2\pi)^{-n} (2i\tau)^{-M} (i\tau)^k}{(M-1)!k!} \int \Psi(\tilde{z}) \mathcal{F}_{z \rightarrow \tilde{z}}[P^M(g_0^k a_\tau)(z)] d\tilde{z},$$

$$\text{with } \Psi(\tilde{z}) = \int_0^1 e^{-\frac{1}{2}i\tau^{-1}t\langle H_z^{-1}\Phi(0)\tilde{z}, \tilde{z} \rangle} (1-t)^{N-1} dt.$$

Since $3k$ derivatives are required to remove the zero of g_0 at $z = 0$, to get a nonzero term in the sum in (4.24) we must have $2j \geq 3k$. Setting $j - k = l$ we have $3l = 3j - 3k \geq j$ and $2l = 2j - 2k \geq k$ so there is just a finite number of terms for each l . Rearranging the terms in the sum in (4.24), we obtain

$$\begin{aligned} \tau^n \int e^{i\tau\Phi(z)} a_\tau(z) dz \\ = e^{i\tau\Phi(0)} \sum_{l=0}^{N-1} \sum_{j-k=l} \sum_{2j \geq 3k} \frac{2^{-j} i^{-l}}{k!j!} \tau^{-l} P^j(g_0^k a_\tau)(0) + R^{(3N)}(\tau) \\ \text{mod } \mathcal{O}(\tau^{m+2N(1-3\rho')}), \end{aligned}$$

as required. □

We conclude this chapter by stating Theorem 4.7 with the notation used at the beginning of the chapter before introducing the $2n$ -dimensional variable z . Notice that since φ is a global phase function, the function $\Phi(x, \xi; y) = \psi(x, y) - \varphi(x, \xi; y)$ has a unique stationary point at $(x, \nabla_y \psi(x, y)|_{y=x}; x)$, when we regard x as a fixed parameter, and the Hessian matrix at the stationary point is

$$\begin{pmatrix} 0 & I \\ I & H_y \Phi(x, \xi_0; y)|_{y=x} \end{pmatrix},$$

with $\xi_0 = \nabla_y \psi(x, y) \Big|_{y=x}$. It follows immediately that the determinant of this matrix and its sign are -1 and 0 respectively.

Theorem 4.8. *Let $a \in \mathcal{S}_{\rho, \delta}^m(\varphi, T^* \mathcal{M})$, $\psi(x; y) \in \mathcal{C}^\infty(\mathcal{M} \times \mathcal{M})$ real valued such that $\nabla_y \psi(x; y) \neq 0$ and $\xi_0 = \nabla_y \psi(x; y) \Big|_{y=x}$. Set $\Phi(x, \xi; y) = \psi(x; y) - \varphi(x, \xi; y)$ and*

$$g_0(x, \xi; y) = \Phi(x, \xi; y) - \Phi(x, \xi_0; x),$$

$$\langle \xi - \xi_0, y - x \rangle - \frac{1}{2} \langle H_y \Phi(x, \xi_0; x)(y - x), y - x \rangle,$$

which vanishes of third order at $(x, \xi; y) = (x, \xi_0; x)$. Then, for all $N \in \mathbb{N}$

$$\begin{aligned} & \int e^{i\tau\psi(x; y)} e^{-i\varphi(x, \xi; y)} p(x, y) \sigma(x, \xi) dy d\xi \\ &= e^{i\tau\psi(x; x)} \sum_{l=0}^{N-1} \sum_{j-k=l} \sum_{2j \geq 3k} \frac{i^{-l} 2^{-j}}{k! j!} \tau^{-l} P^j(g_0^k a_\tau)(x, \xi_0; x) + R^{(3N)}(\tau) \\ & \qquad \qquad \qquad \text{mod } \mathcal{O}(\tau^{m+2N(1-3\rho')}), \end{aligned}$$

with $P = -2\langle \nabla_\xi, \nabla_y \rangle + \langle H_y \Phi(x, \xi_0; x) \nabla_\xi, \nabla_\xi \rangle$, and

$$R^{(3N)}(\tau) = \sum_{k=0}^{2N-1} C_{N,k} \tau^{-3N+k} \int \Psi(\tilde{\xi}; \tilde{y}) \mathcal{F}_{(\xi; y) \rightarrow (\tilde{\xi}; \tilde{y})} [P^{3N}(g_0^k a_\tau)(x, \xi; y)] d\tilde{\xi} d\tilde{y},$$

$$\text{with } \Psi(\tilde{\xi}; \tilde{y}) = \int_0^1 e^{-\frac{1}{2} i \tau^{-1} t [2\langle \tilde{\xi} - \xi_0, \tilde{y} - x \rangle - \langle H_y \Phi(x, \xi_0; x)(\tilde{\xi} - \xi_0), \tilde{\xi} - \xi_0 \rangle]} (1-t)^{N-1} dt.$$

Chapter 5

φ -symbolic Calculus on Manifolds

In this chapter we develop symbolic calculus for φ -symbols. In Section 1.2, we showed that on \mathbb{R}^n there exists a correspondence between the space of symbols and the one of Ψ DOs as $*$ -algebras. We observed that, while it is easily proved that they are isomorphic as vector spaces, the analysis of the multiplicative and involutive structure of these spaces requires a more refined analysis. Consequently, adjoints and compositions of Ψ DOs plays a fundamental role in the theory of symbolic calculus. The main goal is to prove that adjoints and compositions of Ψ DOs as of Definition 3.16 are Ψ DOs of the same kind. In addition, we aim to derive expressions describing their φ -symbols in the form of asymptotic expansions. We exploit the invariant approach presented in Chapter 3 in order to relax the classical conditions $1 - \rho \leq \delta$ and $\rho > 1/2$ arising from the action of change of variables to the less restrictive condition

$$0 \leq \frac{1 - \rho}{2} \leq \delta < \rho \leq 1. \quad (5.1)$$

or even just

$$0 \leq \delta < \rho \leq 1 \quad \wedge \quad \rho > \frac{1}{3}. \quad (5.2)$$

and to extend symbolic calculus to cases otherwise excluded on manifolds. One observes immediately that these are the same conditions required by Safarov in [16] to develop symbolic calculus.

In Section 5.1 we introduce the concept of φ equivalence in the class of φ -amplitudes. In the Euclidean case, one considers to be two amplitudes equivalent defining the same Ψ DO or two Ψ DOs which differ by a smoothing operator. We introduce a similar notion in our new setting. As we saw in Section 3.3 the representation of a Ψ DO in $L_{\rho,\delta}^m(\varphi, \mathcal{M}, \Omega^{1/2})$ depends on the choice of a global phase function φ . Therefore, our notion of equivalence will depend on the phase function. Moreover, for a fixed global phase function φ , we show how to obtain an asymptotic expansion for the φ -symbol $\sigma_{\varphi,A}(x, \xi)$ of a Ψ DO A in terms of one of its φ -amplitudes $a(x, \xi; y)$. We derive a formula which maintains the same simplicity and elegance of the corresponding formula in the Euclidean context. The main difference is the presence of non-constant coefficients. It holds under the usual assumption that $0 \leq \delta < \rho \leq 1$.

In Section 5.2, we present the central result of this work. We prove that the composition of two Ψ DOs $A \in L_{\rho,\delta}^{m_1}(\varphi, \mathcal{M}, \Omega^{1/2})$ and $B \in L_{\rho,\delta}^{m_2}(\varphi, \mathcal{M}, \Omega^{1/2})$ is in $L_{\rho,\delta}^{m_1+m_2}(\varphi, \mathcal{M}, \Omega^{1/2})$ and the adjoint A^* is in $L_{\rho,\delta}^{m_1}(\varphi, \mathcal{M}, \Omega^{1/2})$. In addition, we derive the asymptotic expansion describing the symbols σ_{AB} and σ_{A^*} . We discuss the operator AB^* from which the results for A^* and AB follows as a corollary. The series describing the symbol σ_{AB^*} is obtained by means of the stationary phase formula for oscillatory integrals derived in Theorem 4.8. This expression involves the use of the curved derivative $\mathfrak{D}_{\varphi,x}^\alpha$ introduced in Section 3.1. Furthermore, the coefficients in this formula are non-constant and depend on the choice of global

phase function. Finally, we also need to assume condition (5.1) so that the type of the operator is preserved. Alternatively, sacrificing something on the type of the operators involved, one can simply assume (5.2). The asymptotic formula derived in Section 5.2 has an evident correspondence with formula (2.20) obtained by Safarov in [17, Section 8].

From now on we assume the following notation:

$$\delta'' = \max \left\{ \delta, \frac{1-\rho}{2} \right\}.$$

5.1 φ -equivalence of Amplitudes

In the local coordinates case we noticed that the map that associates an amplitude with a Ψ DO is highly not injective, while it is essentially bijective if we restrict the domain to the class of symbols. In other words, given an amplitude, we can always associate a symbol defining the same Ψ DO modulo a smoothing operator. Proposition 1.15 gives us an account of this technique. This section is devoted to replicating this result in the new context introduced in Chapter 3.

The standard results in local coordinates concerning how to obtain from a given amplitude an equivalent symbol rely on Taylor's formula. They use integration by parts and the equality

$$(x_j - y_j)e^{i\langle x-y, \xi \rangle} = -i\partial_{\xi_j} e^{i\langle x-y, \xi \rangle}. \quad (5.3)$$

Because of the global phase function φ in the representation, some additional attention in handling Taylor's expansion is required. It all comes down to noticing that we need to introduce a suitable function $h(x, \xi; y)$, which is positively homogeneous

of degree 0 in ξ , in order to adjust the formula (5.3) to the new setting:

$$(x_j - y_j)h(x, \xi; y)e^{i\varphi(x, \xi; y)} = \partial_{\xi_j} e^{i\varphi(x, \xi; y)}.$$

This fact produces minor modifications of the classical formulae. We emphasise that all the new results are consistent with the classical ones, which can be retrieved simply by setting $\varphi(x, \xi; y) = \langle x - y, \xi \rangle$.

We recall that, as a rule, oscillatory integrals are used only for studying the singularities of distributions and hence all calculations are carried out modulo $\mathcal{S}^{-\infty}(T^*\mathcal{M} \times \mathcal{M})$.

It is clear from Section 1.1 that kernel of a Ψ DO

$$\int e^{i\varphi(x, \xi; y)} a(x, \xi; y) d\xi, \quad a \in \mathcal{S}_{\rho, \delta}^m(\varphi, T^*\mathcal{M} \times \mathcal{M}),$$

is determined modulo $\mathcal{C}^\infty(\mathcal{M} \times \mathcal{M})$ by the behaviour of a for large ξ on a small conic neighbourhood of the diagonal in $\mathcal{M} \times \mathcal{M}$. Since we disregard smoothing operators, we can assume a to be nonzero only for sufficiently large ξ and near the diagonal in $\mathcal{M} \times \mathcal{M}$.

Furthermore, similarly to the Euclidean case, we express the φ -symbol $\sigma_{\varphi, A}$ of a Ψ DO A not by an exact formula but by an asymptotic expansion, which is sufficient for the analysis of singularities. The following definition is a repetition of Definition 1.13 adjusted to the new context.

Definition 5.1. Let $a_j \in \mathcal{S}_{\rho, \delta}^{m_j}(\varphi, T^*\mathcal{M} \times \mathcal{M})$, where $m_j \in \mathbb{R}$ and $m_j \rightarrow -\infty$, and $a \in \mathcal{S}_{\rho, \delta}^{m_0}(\varphi, T^*\mathcal{M} \times \mathcal{M})$. We write

$$a(x, \xi; y) \sim \sum_{j=0}^{\infty} a_j(x, \xi; y)$$

and we say that $\sum_{j=0}^{\infty} a_j$ is an asymptotic expansion of a , if for any integer $N \geq 0$ we have

$$a(x, \xi; y) - \sum_{j=0}^N a_j(x, \xi; y) \in \mathcal{S}_{\rho, \delta}^{l_N}(\varphi, T^*\mathcal{M} \times \mathcal{M}),$$

with $l_N \rightarrow -\infty$.

We shall make frequent use of the following lemma. This is a simple modification of [3, Theorem 8.16] and it is proved in a similar fashion.

Lemma 5.2. *Let $a_j \in \mathcal{S}_{\rho, \delta}^{m_j}(\varphi, T^*\mathcal{M} \times \mathcal{M})$ where $m_j \rightarrow -\infty$. There exists a function $a \in \mathcal{S}_{\rho, \delta}^{m_0}(\varphi, T^*\mathcal{M} \times \mathcal{M})$ unique modulo $S^{-\infty}(T^*\mathcal{M} \times \mathcal{M})$, such that $a \sim \sum_j a_j$.*

Proof. Choose $\chi \in \mathcal{C}^\infty(T^*\mathcal{M})$ such that $\chi(x, \xi) = 1$ for $\langle \xi \rangle_x \geq 1$ and $\chi(x, \xi) = 0$ for $\langle \xi \rangle_x \leq 1/2$. Possibly rearranging the terms of the series $\sum_j a_j$, assume $m_j \searrow -\infty$.

We claim that there is a sequence $\{t_j\}_j$ of positive numbers converging to zero such that for each $j \geq 1$ and $|\alpha| + |\beta| \leq j$

$$|\partial_\xi^\alpha \mathfrak{D}_{\varphi, x}^\beta [\chi(x, t_j \xi) a_j(x, \xi; y)]| \leq 2^{-j} \langle \xi \rangle_x^{m_{j-1} - \rho|\alpha| + \delta|\beta|}. \quad (5.4)$$

In fact, observe that $\chi(x, t\xi) \in \mathcal{S}_{1,0}^0(T^*\mathcal{M})$ uniformly for $t \leq 1$. Therefore, $\chi(x, t\xi) a_j(x, \xi; y) \in \mathcal{S}_{\rho, \delta}^{m_j}(T^*\mathcal{M} \times \mathcal{M})$ and there exist $C_j > 0$ independent of t such that for all $|\alpha| + |\beta| \leq j$, $t \leq 1$ one has

$$|\partial_\xi^\alpha \mathfrak{D}_{\varphi, x}^\beta \chi(x, t_j \xi) a_j(x, \xi; y)| \leq C_j \langle \xi \rangle_x^{m_j - \rho|\alpha| + \delta|\beta|}.$$

Observe that the expression on the left is actually zero if $\langle \xi \rangle_x \leq (2t)^{-1}$. Therefore to show that the claim holds, it suffices to choose t_j small enough so that

- $t_j \rightarrow 0$;
- if $\langle \xi \rangle_x \geq (2t_j)^{-1}$, then $C_j \langle \xi \rangle_x^{m_j - m_{j-1}} \leq 2^{-j}$.

Next, set

$$a(x, \xi; y) = \sum_{j=0}^{\infty} \chi(x, t_j \xi) a_j(x, \xi; y).$$

Since $t_j \rightarrow 0$, there are only finitely many nonzero terms in this sum and it follows that $a \in \mathcal{C}^\infty(T^*\mathcal{M} \times \mathcal{M})$. We wish to estimate $\partial_\xi^\alpha \mathfrak{D}_{\varphi, x}^\beta a(x, \xi; y)$. Choose N large enough so that $|\alpha| + |\beta| \leq N$ and write

$$a(x, \xi; y) = \sum_{j=0}^N \chi(x, t_j \xi) a_j(x, \xi; y) + \sum_{j=N+1}^{\infty} \chi(x, t_j \xi) a_j(x, \xi; y).$$

Since $a_j \in \mathcal{S}_{\rho, \delta}^{m_j}(\varphi, T^*\mathcal{M} \times \mathcal{M})$ and $\chi(x, t_j \xi) = 1$ for $\langle \xi \rangle_x$ large, we clearly have

$$\left| \sum_{j=0}^N \partial_\xi^\alpha \mathfrak{D}_{\varphi, x}^\beta \chi(x, t_j \xi) a_j(x, \xi; y) \right| \leq \sum_{j=0}^N C_j \langle \xi \rangle_x^{m_j - \rho|\alpha| + \delta|\beta|} \leq C \langle \xi \rangle_x^{m_0 - \rho|\alpha| + \delta|\beta|}.$$

On the other hand, by (5.4) we have

$$\left| \sum_{j=N+1}^{\infty} \partial_\xi^\alpha \mathfrak{D}_{\varphi, x}^\beta \chi(x, t_j \xi) a_j(x, \xi; y) \right| \leq \sum_{j=N+1}^{\infty} 2^{-j} \langle \xi \rangle_x^{m_{j-1} - \rho|\alpha| + \delta|\beta|} \leq C \langle \xi \rangle_x^{m_0 - \rho|\alpha| + \delta|\beta|}.$$

It follows that $a \in \mathcal{S}_{\rho, \delta}^{m_0}(\varphi, T^*\mathcal{M} \times \mathcal{M})$. Using the same argument one can show that $\sum_{j=N+1}^{\infty} \partial_\xi^\alpha \mathfrak{D}_{\varphi, x}^\beta \chi(x, t_j \xi) a_j(x, \xi; y) \in \mathcal{S}_{\rho, \delta}^{m_N}$ for all $N \in \mathbb{N}$. Furthermore, since $\chi(x, t_j \xi) a_j(x, \xi; y) = a_j(x, \xi; y)$ for ξ large, it follows that $a \sim \sum a_j$. \square

The notion of asymptotic expansion and Lemma 5.2 presented above allows to introduce also in our new setting the concept of classical amplitudes and classical symbols.

Definition 5.3. We denote by $\mathcal{S}^m(T^*\mathcal{M} \times \mathcal{M})$ the class of amplitudes $a \in \mathcal{S}_{1,0}^m(T^*\mathcal{M} \times \mathcal{M})$ that admits an asymptotic expansion

$$a(x, \xi; y) \sim \sum_{j=0}^{\infty} a_{m-j}(x, \xi; y),$$

with a_{m-j} positively homogeneous in ξ of degree $m - j$. If a is independent of y , it is called the classical symbol and we denote this class by $\mathcal{S}^m(T^*\mathcal{M})$. In this case the leading homogeneous term a_m is called the principal symbol.

As usual operators with smooth kernel are negligible. It is therefore useful to introduce an equivalence relation in the class of amplitudes. This relation associates two φ -amplitudes whose difference generates an operator with smooth kernel. Notice that this equivalence relation depends on the choice of a global phase function φ .

Definition 5.4. Let φ be global phase functions, a, b two φ -amplitudes and

$$\mathcal{K}_{a,b}(x, y) = \mathbf{p}(x; y) \int e^{i\varphi(x, \xi; y)} [a(x, \xi; y) - b(x, \xi; y)] d\xi$$

We say that a, b are φ -equivalent, and denote by $a \sim_{\varphi} b$, if and only if $\mathcal{K}_{a,b}(\cdot, y)$ and $\mathcal{K}_{a,b}(x, \cdot)$ are both elements of $\mathcal{C}^{\infty}(\mathcal{M}, \Omega^{1/2})$.

We are now ready to present the formula corresponding to (5.3) adjusted for Ψ DOs represented by means of a global phase function φ discussed above.

Lemma 5.5. *Let a be a φ -amplitude in $\mathcal{S}_{\rho,\delta}^m(\varphi, T^*\mathcal{M} \times \mathcal{M})$. If the coordinates $\{y^k\}$ coincide with $\{x^k\}$, then for any multi-index α*

$$(x - y)^{\alpha} a(x, \xi; y) \sim_{\varphi} \sum_{|\nu| \leq |\alpha|} p_{\varphi, \alpha, \nu}(x, \xi; y) \partial_{\xi}^{\nu} a(x, \xi; y),$$

where $p_{\varphi, \alpha, \nu}$ is positively homogeneous of degree $|\nu| - |\alpha|$ in ξ .

Proof. First, we consider the case $|\alpha| = 1$. Assume $(x - y)^\alpha = x_k - y_k$. Formula (3.5) implies that for x and y nearby

$$x_k - y_k = \sum_{j=1}^n \tilde{p}_{\varphi,k,j}(x, \xi; y) \partial_{\xi_j} \varphi(x, \xi; y), \quad (5.5)$$

with $\tilde{p}_{\varphi,j,k}$ positively homogeneous of degree 0 in ξ . Using (5.5) and integrating by parts in

$$\int e^{i\varphi(x,\xi;y)} (x_k - y_k) a(x, \xi; y) d\xi,$$

using formula $e^{i\varphi} \partial_{\xi_k} \varphi = -i \partial_{\xi_k} e^{i\varphi}$, we obtain

$$\begin{aligned} & (x_k - y_k) a(x, \xi; y) \\ & \sim_{\varphi} \sum_{j=1}^n \tilde{p}_{\varphi,k,j}(x, \xi; y) \partial_{\xi_j} a(x, \xi; y) + \partial_{\xi_j} \tilde{p}_{\varphi,k,j}(x, \xi; y) a(x, \xi; y) \\ & = \sum_{|\nu| \leq |\alpha|=1} p_{\varphi,\alpha,\nu}(x, \xi; y) \partial_{\xi}^{\nu} a(x, \xi; y) \end{aligned}$$

where $p_{\varphi,\alpha,\nu}$ is positively homogeneous of degree $|\alpha| - |\nu| = 1 - |\nu|$ in ξ . This concludes the proof for $|\alpha| = 1$.

If $|\alpha| > 1$ we just need to repeat the process above sufficiently many times. \square

Lemma 5.5 introduces the key result of this section. The following theorem gives us a formula for the φ -symbol of a Ψ DO A whose kernel is

$$\mathcal{A}_{\varphi}(x; y) = \mathbf{p}(x, y) \int e^{i\varphi(x,\xi;y)} a(x, \xi; y) d\xi, \quad a \in \mathcal{S}_{\rho,\delta}^m(\varphi, T^*\mathcal{M} \times \mathcal{M}).$$

The main difference with the corresponding result in local coordinates (see Proposition 1.15) is that the asymptotic expansion describing the symbol has non-constant

coefficients. However, the technique remains valid under the usual assumption $0 \leq \delta < \rho \leq 1$. The proof is similar. In fact, it is once again based on Taylor's formula and integration by parts. Calculations require some care because the global phase function φ replaces $\langle x - y, \xi \rangle$, as noted at the beginning of this section.

Theorem 5.6. *Let a be a φ -amplitude in $\mathcal{S}_{\rho,\delta}^m(\varphi, T^*\mathcal{M} \times \mathcal{M})$. Then,*

$$a(x, \xi; y) \underset{\varphi}{\sim} \sum_{\alpha, \beta} p_{\varphi, \alpha, \beta}(x, \xi) \partial_{\xi}^{\alpha} \partial_y^{\beta} a(x, \xi; y) \Big|_{y=x}, \quad (5.6)$$

where $p_{\varphi, \alpha, \beta} \in \mathcal{S}^{\min\{|\alpha| - |\beta|, 0\}}(T^*\mathcal{M})$. In particular, $p_{\varphi, 0, 0} = 1$ and therefore

$$a(x, \xi; y) \underset{\varphi}{\sim} a(x, \xi; y) \Big|_{y=x} \quad \text{mod } \mathcal{S}_{\rho,\delta}^{m-\rho+\delta}(\varphi, T^*\mathcal{M}). \quad (5.7)$$

Remark 5.7. It follows immediately from Theorem 5.6 that if $A \in L_{\rho,\delta}^m(\varphi, \mathcal{M}, \Omega^{1/2})$, there exists a φ -symbol, which will be denoted by $\sigma_{\varphi, A}$, in $\mathcal{S}_{\rho,\delta}^m(\varphi, T^*\mathcal{M})$ unique modulo $\mathcal{S}^{-\infty}(T^*\mathcal{M})$ such that

$$Au(x) = \int e^{i\varphi(x, \xi; y)} \mathbf{p}(x, y) \sigma_{\varphi, A}(x, \xi) u(y) dy d\xi, \quad u \in \mathcal{C}^{\infty}(\mathcal{M}, \Omega^{1/2}).$$

The proof of Theorem 5.6 is a consequence of the following two lemmas.

Lemma 5.8. *Let $c(x, \xi; y) \in \mathcal{C}^{\infty}(T^*\mathcal{M} \times \mathcal{M})$ be positively homogeneous of degree k in ξ and $\sigma(x, \xi) \in \mathcal{S}_{\rho,\delta}^m(\varphi, T^*\mathcal{M})$. Then,*

$$c(x, \xi; y) \sigma(x, \xi) \underset{\varphi}{\sim} \sum_{\alpha} c_{\varphi, \alpha}(x, \xi) \partial_{\xi}^{\alpha} \sigma(x, \xi), \quad (5.8)$$

with $c_{\varphi, \alpha} \in \mathcal{S}^k(T^*\mathcal{M})$.

Proof. By Taylor's formula, we have

$$c(x, \xi; y)\sigma(x, \xi) = c(x, \xi; y)\Big|_{y=x} \sigma(x, \xi) + R_1(x, \xi; y),$$

where

$$R_1(x, \xi; y) = \sum_{k=1}^n (y_k - x_k)\sigma(x, \xi) \int_0^1 \partial_{y_k} c(x, \xi; y)\Big|_{y=y_t} dt,$$

with $y_t = x + t(y - x)$. Let $c_{\varphi, \mathbf{0}}^{(0)}(x, \xi) = c(x, \xi; y)\Big|_{y=x}$. By Lemma 5.5, we obtain

$$c(x, \xi; y)\sigma(x, \xi) \sim_{\varphi} c_{\varphi, \mathbf{0}}^{(0)}(x, \xi)\sigma(x, \xi) + R_1(x, \xi; y),$$

where

$$R_1(x, \xi; y) = \sum_{|\alpha| \leq 1} c_{\varphi, \alpha}^{(1)}(x, \xi; y) \partial_{\xi}^{\alpha} \sigma(x, \xi),$$

with $c_{\varphi, \alpha}^{(1)}(x, \xi; y)$ positively homogeneous of degree $k + |\alpha| - 1$. We notice that $c_{\varphi, \alpha}^{(1)}(x, \xi; y) \partial_{\xi}^{\alpha} \sigma(x, \xi)$ is still the product of a positively homogeneous function in ξ and a φ -symbol, but the order of R_1 is at most $m + k - \rho$. Therefore, we may repeat the same process on R_1 . Let $c_{\varphi, \alpha}^{(1)}(x, \xi) = c_{\varphi, \alpha}^{(1)}(x, \xi; y)\Big|_{y=x}$, then

$$R_1(x, \xi; y) \sim_{\varphi} \sum_{|\alpha| \leq 1} c_{\varphi, \alpha}^{(1)}(x, \xi) \partial_{\xi}^{\alpha} \sigma(x, \xi) + R_2(x, \xi; y),$$

where

$$R_2(x, \xi; y) = \sum_{|\alpha'| \leq 2} c_{\varphi, \alpha'}^{(2)}(x, \xi; y) \partial_{\xi}^{\alpha'} \sigma(x, \xi),$$

with $c_{\varphi, \alpha'}^{(2)}(x, \xi; y)$ positively homogeneous of degree $k + |\alpha'| - 2$. The order of R_2 is at most $m + k - 2\rho$. Repeating this process we obtain an asymptotic expansion.

Rearranging the terms according to α , we obtain (5.8) where

$$c_{\varphi,\alpha}(x, \xi) = \sum_{j=|\alpha|}^{\infty} c_{\varphi,\alpha}^{(j)}(x, \xi) \in \mathcal{S}^k(T^*\mathcal{M}).$$

□

Lemma 5.9. *Let a be a φ -amplitude in $\mathcal{S}_{\rho,\delta}^m(\varphi, T^*\mathcal{M} \times \mathcal{M})$. Then*

$$a(x, \xi; y) \underset{\varphi}{\sim} \sum_{|\alpha| \leq |\beta|} q_{\varphi,\alpha,\beta}(x, \xi; y) \partial_{\xi}^{\alpha} \partial_y^{\beta} a(x, \xi; y) \Big|_{y=x},$$

where $q_{\varphi,\alpha,\beta}$ is positively homogeneous of degree $|\alpha| - |\beta|$ in ξ . In particular, $q_{\varphi,0,0} = 1$.

Proof. By Taylor's formula, we have

$$a(x, \xi; y) = \sum_{|\beta| \leq N} \frac{1}{\beta!} (y - x)^{\beta} \partial_y^{\beta} a(x, \xi; y) \Big|_{y=x} + R_N(x, \xi; y),$$

where

$$R_N(x, \xi; y) = (N + 1) \sum_{|\beta|=N+1} \frac{1}{\beta!} (y - x)^{\beta} \int_0^1 (1 - t)^N \partial_y^{\beta} a(x, \xi; y) \Big|_{y=y_t} dt,$$

with $y_t = x + t(y - x)$. By Lemma 5.5, we obtain

$$a(x, \xi; y) \underset{\varphi}{\sim} \sum_{|\alpha| \leq |\beta| \leq N} q_{\varphi,\alpha,\beta}(x, \xi; y) \partial_{\xi}^{\alpha} \partial_y^{\beta} a(x, \xi; y) \Big|_{y=x} + \tilde{R}_N(x, \xi; y),$$

where $q_{\varphi,\alpha,\beta}(x, \xi; y)$ is a positively homogeneous function of degree $|\alpha| - |\beta|$ in ξ , $q_{\varphi,0,0}(x, \xi; y) = 1$ and

$$\tilde{R}_N(x, \xi; y) = \sum_{|\alpha| \leq |\beta|=N+1} q_{\varphi,\alpha,\beta}(x, \xi; y) \int_0^1 (1 - t)^N \partial_{\xi}^{\alpha} \partial_y^{\beta} a(x, \xi; y) \Big|_{y=y_t} dt.$$

We observe that

$$|\tilde{R}_N(x, \xi; y)| \leq C \langle \xi \rangle_x^{m-(N+1)(\rho-\delta)}.$$

Therefore, we conclude that

$$\begin{aligned} & \mathfrak{p}(x; y) \int e^{i\varphi(x, \xi; y)} \left[a(x, \xi; y) \right. \\ & \quad \left. - \sum_{|\alpha| \leq |\beta|} q_{\varphi, \alpha, \beta}(x, \xi; y) \partial_\xi^\alpha \partial_y^\beta a(x, \xi; y) \Big|_{y=x} \right] d\xi \in \mathcal{C}^\infty(\mathcal{M} \times \mathcal{M}), \end{aligned}$$

as required. \square

Proof (Theorem 5.6). By Lemma 5.9,

$$a(x, \xi; y) \underset{\varphi}{\sim} \sum_{|\alpha| \leq |\beta|} q_{\varphi, \alpha, \beta}(x, \xi; y) \partial_\xi^\alpha \partial_y^\beta a(x, \xi; y) \Big|_{y=x}.$$

where $q_{\varphi, \alpha, \beta}$ is positively homogeneous of degree $|\alpha| - |\beta|$ in ξ and $q_{\varphi, \mathbf{0}, \mathbf{0}} = 1$. In addition, by Lemma 5.8,

$$q_{\varphi, \alpha, \beta}(x, \xi; y) \partial_\xi^\alpha \partial_y^\beta a(x, \xi; y) \Big|_{y=x} \underset{\varphi}{\sim} \sum_{\nu} c_{\varphi, \alpha, \nu, \beta}(x, \xi) \partial_\xi^{\alpha+\nu} \partial_y^\beta a(x, \xi; y) \Big|_{y=x}, \quad (5.9)$$

where $c_{\varphi, \alpha, \nu, \beta} \in \mathcal{S}^{|\alpha| - |\beta|}(T^*\mathcal{M})$. Rearranging the series on the right hand side of (5.9), we obtain (5.6). In addition, (5.7) follows immediately since $c_{\varphi, \mathbf{0}, \mathbf{0}, \mathbf{0}} = 1$. \square

Next, we consider the identity operator I . It is evident that for a general global phase function φ the identity operator cannot be the constant function identically 1. However, $\sigma_{\varphi, I}$ has some nice properties. In particular, we prove that $\sigma_{\varphi, I}$ is a classical symbol of order zero and has principal symbol identically 1.

Proposition 5.10. *Let φ be a global phase function. Then there exists a symbol $\sigma_{\varphi,I} \in \mathcal{S}^0(T^*\mathcal{M})$ such that the kernel*

$$\mathbf{p}(x; y) \int e^{i\varphi(x, \xi; y)} \sigma_{\varphi,I}(x, \xi) u(y) dy d\xi - u(x) \underset{\varphi}{\sim} Ku(x),$$

where K is an operator with smooth kernel acting in the space of $1/2$ -densities.

In addition,

$$\sigma_{\varphi,I}(x, \xi) = 1 \quad \text{mod } \mathcal{S}^{-1}(T^*\mathcal{M}). \quad (5.10)$$

Proof. By Theorem 2.5, there exists a symbol $\sigma_{\varphi,I} \in \mathcal{S}^0(T^*\mathcal{M})$ such that the oscillatory integral

$$\mathbf{p}(x; y) \int e^{i\varphi(x, \xi; y)} \sigma_{\varphi,I}(x, \xi) d\xi,$$

is the Schwartz kernel of the identity operator I modulo \mathcal{C}^∞ . Let $\{x^k\}$ and $\{y^k\}$ be the same local coordinates, set

$$\mathcal{I}(x; y) = \mathbf{p}(x; y) \int e^{i\langle x-y, \xi \rangle} 1 d\xi.$$

Since $\mathbf{p}(x; x) = 1$ for all $x \in \mathcal{M}$, we observe that for any $u \in \mathcal{C}^\infty(\mathcal{M}, \Omega^{1/2})$

$$\int \mathcal{I}(x; y) u(y) dy = u(x).$$

For x and y sufficiently close, using the the same transformation used to obtain (3.28), we have

$$\mathcal{I}(x; y) = \mathbf{p}(x; y) \int e^{i\varphi(x, \xi; y)} |\det\{d\tilde{\xi}^k/d\xi^j\}| d\xi,$$

with $\tilde{\xi} = \tilde{\xi}(x, \xi; y) = \xi + V(x, \xi; y)$, where $V(x, \xi; y)$ is a vector-valued function with positively homogeneous of degree 1 in ξ entries and $V(x, \xi; y) = 0$ at $y = x$. Notice

that in a neighbourhood of $y = x$, $|\det\{d\tilde{\xi}^k/d\xi^j\}| = \det\{d\tilde{\xi}^k/d\xi^j\}$. By Theorem 5.6

$$\det\{d\tilde{\xi}^k/d\xi^j\} \underset{\varphi}{\sim} 1 + \sum_{\alpha>0, \beta>0} p_{\varphi, \alpha, \beta}(x, \xi) \partial_{\xi}^{\alpha} \partial_y^{\beta} \det\{d\tilde{\xi}^k/d\xi^j\} \Big|_{y=x},$$

with $p_{\varphi, \alpha, \beta} \in \mathcal{S}^{\min\{|\alpha|-|\beta|, 0\}}(T^*\mathcal{M})$. This implies immediately (5.10). \square

We conclude this section with a conjecture regarding the relation between the spaces $\mathcal{S}_{\rho, \delta}^m(\varphi, T^*\mathcal{M})$ and $L_{\rho, \delta}^m(\varphi, \mathcal{M}, \Omega^{1/2})$. In the Euclidean case, one has an isomorphic correspondence between the factor class $\mathcal{S}_{\rho, \delta}^m(X \times \mathbb{R}^n)/\mathcal{S}^{-\infty}(X \times \mathbb{R}^n)$ and $L_{\rho, \delta}^m(X)/L_{\rho, \delta}^{-\infty}(X)$ as vector spaces. We conjecture that an equivalent result can be achieved under the condition $1 - \rho \leq 2\delta$ for the classes $\mathcal{S}_{\rho, \delta}^m(\varphi, T^*\mathcal{M})$ and $L_{\rho, \delta}^m(\varphi, \mathcal{M}, \Omega^{1/2})$.

Conjecture. Let $1 - \rho \leq 2\delta$. There exists an isomorphism of vector spaces between factor classes $L_{\rho, \delta}^m(\varphi, \mathcal{M}, \Omega^{1/2})/L^{-\infty}(\mathcal{M}, \Omega^{1/2})$ and $\mathcal{S}_{\rho, \delta}^m(\varphi, T^*\mathcal{M})/\mathcal{S}^{-\infty}(T^*\mathcal{M})$.

5.2 Composition Formula

This section contains the main original result of the work. We prove that the adjoint of a Ψ DO A from the class $L_{\rho, \delta}^{m_1}(\varphi, \mathcal{M}, \Omega^{1/2})$ and the composition between A and a second Ψ DO B from the class $L_{\rho, \delta}^{m_2}(\varphi, \mathcal{M}, \Omega^{1/2})$ are two Ψ DOs A^* and AB from the classes $L_{\rho, \delta}^{m_1}(\varphi, \mathcal{M}, \Omega^{1/2})$ and $L_{\rho, \delta}^{m_1+m_2}(\varphi, \mathcal{M}, \Omega^{1/2})$ respectively. These results hold in general under the condition (5.1). This requirement can be relaxed to (5.2) sacrificing the invariance of the type of the resulting operator. In fact, in this case the adjoint A^* and the composition AB are elements of the classes $L_{\rho, \delta''}^{m_1}(\varphi, \mathcal{M}, \Omega^{1/2})$ and $L_{\rho, \delta''}^{m_1+m_2}(\varphi, \mathcal{M}, \Omega^{1/2})$ respectively. The proofs of these results rely on the stationary phase formula for oscillatory integrals proved in Theorem

4.8. The asymptotic expansions describing the φ -symbols of the adjoint and the composition feature non-constant coefficients. However, it has a similar structure to the composition formula in local coordinates (see Proposition 1.18). We choose to study the operator AB^* and obtain the classical results for the adjoint operator A^* and the composition operator AB as a corollary.

Now, we introduce some preliminary results necessary for the proof of the general theorem on AB^* . The main purpose of these is to investigate the behaviour of a function of the form $a(y, -\nabla_y \varphi(x, \xi; y))$, where $a \in \mathcal{S}_{\rho, \delta}^m(\varphi, T^*\mathcal{M})$ and φ is a global phase function. This function appears naturally when studying σ_{φ, AB^*} by means of the stationary phase formula. It will become clear that the operator $\mathfrak{D}_{\varphi, x}^\alpha$ defined in (3.7) is modelled on this function.

The following result establishes a connection between the usual derivatives d^β/dy^β and $\mathfrak{D}_{\varphi, x}^\beta$. Because of the relevance to our study, the lemma highlights the presence of factors vanishing on the diagonal \mathcal{D} in $\mathcal{M} \times \mathcal{M}$ of the homogeneous coefficients $h_{\beta, \nu, \mu}(x, \xi; y)$. This is crucial for our purposes. We observe immediately that in general one only has

$$\left| \frac{d^\beta}{dy^\beta} a(y, -\nabla_y \varphi(x, \xi; y)) \right| \leq C \langle \xi \rangle_x^{m + \delta' |\beta|}.$$

Nevertheless, integrating by parts the factors vanishing on the diagonal \mathcal{D} we prove that $d^\beta/dy^\beta a(y, -\nabla_y \varphi(x, \xi; y))$ is φ -equivalent to an element in $\mathcal{S}_{\rho, \delta'}^{m + \delta'' |\beta|}(\varphi, T^*\mathcal{M} \times \mathcal{M})$.

Lemma 5.11. *Let $a(x, \xi) \in \mathcal{C}^\infty(T^*\mathcal{M})$. Then,*

$$\frac{d^\beta}{dy^\beta} a(y, -\nabla_y \varphi(x, \xi; y)) = \sum_{|\mu| + |\nu| \leq |\beta|} h_{\beta, \nu, \mu}(x, \xi; y) (\mathfrak{D}_{\varphi, x}^\mu \partial_\xi^\nu a)(y, -\nabla_y \varphi(x, \xi; y)), \quad (5.11)$$

where $h_{\beta,\nu,\mu}(x, \xi; y)$ is positively homogeneous of degree $|\nu|$ in ξ with a zero at $y = x$ of order $\max\{2|\nu| + |\mu| - |\beta|, 0\}$.

Proof. We prove this by induction on $|\beta|$. The case $|\beta| = 1$ is an immediate consequence of Lemma 3.14. As for the general case, again by Lemma 3.14 we have

$$\begin{aligned} \frac{d^{\beta+e_k}}{dy^{\beta+e_k}} a(y, -\nabla_y \varphi(x, \xi; y)) &= \frac{d^\beta}{dy^\beta} \left[(\mathfrak{D}_{\varphi,x}^{e_k} a)(y, -\nabla_y \varphi(x, \xi; y)) \right. \\ &\quad \left. + \sum_{j=1}^n m_{k,j}(x, \xi; y) (\partial_{\xi_j} a)(y, -\nabla_y \varphi(x, \xi; y)) \right], \end{aligned}$$

where $m_{k,j}$ are positively homogeneous of degree 1 in ξ and vanishing on the diagonal \mathcal{D} . By induction hypothesis, (3.21) and (3.24) imply the desired result. \square

Corollary 5.12. *Let $a(x, \xi) \in \mathcal{S}_{\rho,\delta}^m(\varphi, T^*\mathcal{M})$. There exists $\tilde{a} \in \mathcal{S}_{\rho,\delta'}^{m+\delta''|\beta|}(\varphi, T^*\mathcal{M} \times \mathcal{M})$ such that*

$$\frac{d^\beta}{dy^\beta} a(y, -\nabla_y \varphi(x, \xi; y)) \underset{\varphi}{\sim} \tilde{a}(x, \xi; y).$$

Proof. We consider formula (5.11) and we turn the factors vanishing at $y = x$ contained in $h_{\beta,\nu,\mu}(x, \xi; y)$ into differentiation by ξ according to the procedure described in Lemma 5.5. If $2|\nu| + |\mu| \geq |\beta|$, we obtain a φ -equivalent φ -amplitude of order

$$m + \max_{|\nu|+|\mu|\leq|\beta|} \{|\nu| - \rho|\nu| + \delta|\mu| - 2\rho|\nu| - \rho|\mu| + \rho|\beta|\} \leq m + \delta''|\beta|.$$

If $2|\nu| + |\mu| \leq |\beta|$, we obtain a φ -equivalent φ -amplitude of order

$$m + \max_{|\nu|+|\mu|\leq|\beta|} \{|\nu| - \rho|\nu| + \delta|\mu|\} \leq m + \delta''|\beta|.$$

\square

Since $a(y, -\nabla_y \varphi(x, \xi; y)) \in \mathcal{S}_{\rho, \delta'}^m(\varphi, T^*\mathcal{M} \times \mathcal{M})$, one might think that by means of Theorem 5.6 we obtain a φ -equivalent symbol only under the classical condition $1 - \rho \leq \delta$. However, by means of Corollary 5.12, one can obtain an asymptotic expansion for the φ -symbol under the less restrictive requirement $1 - \rho \leq 2\delta$. The following proposition shows it. The outline of the proof is the same as the one of Proposition 5.6. It consists essentially of considering Taylor's expansion of $a(y, -\nabla_y \varphi(x, \xi; y))$ at $y = x$. The condition $1 - \rho \leq 2\delta$ is required to control the order the remainder term in Taylor's expansion of $a(y, -\nabla_y \varphi(x, \xi; y))$.

Proposition 5.13. *Let $a \in \mathcal{S}_{\rho, \delta}^m(\varphi, T^*\mathcal{M})$. If $1 - \rho \leq 2\delta$, then*

$$a(y, -\nabla_y \varphi(x, \xi; y)) \underset{\varphi}{\sim} \sum_{\alpha, \beta} p_{\varphi, \alpha, \beta}(x, \xi) \partial_{\xi}^{\alpha} \mathfrak{D}_{\varphi, x}^{\beta} a(x, \xi), \quad (5.12)$$

where $p_{\varphi, \alpha, \beta} \in \mathcal{S}^{\min\{|\alpha| - |\beta|, 0\}}(T^*\mathcal{M})$. In particular, $p_{\varphi, \mathbf{0}, \mathbf{0}} = 1$ and therefore

$$a(y, -\nabla_y \varphi(x, \xi; y)) \underset{\varphi}{\sim} a(x, \xi) \quad \text{mod } \mathcal{S}_{\rho, \delta}^{m - \rho + \delta}(\varphi, T^*\mathcal{M}).$$

Proof. First, we prove an analogous version of Lemma 5.9 in the new setting. In fact, we claim

$$a(y, -\nabla_y \varphi(x, \xi; y)) \underset{\varphi}{\sim} \sum_{|\alpha| \leq |\beta|} q_{\varphi, \alpha, \beta}(x, \xi; y) \partial_{\xi}^{\alpha} \mathfrak{D}_{\varphi, x}^{\beta} a(x, \xi), \quad (5.13)$$

where $q_{\varphi, \alpha, \beta}$ is positively homogeneous of degree $|\alpha| - |\beta|$ in ξ and $q_{\varphi, \mathbf{0}, \mathbf{0}} = 1$.

We expand $a(y, -\nabla_y \varphi(x, \xi; y))$ at $y = x$ by Taylor's formula and turning the factors vanishing at $y = x$ into differentiation in ξ , according to the procedure described in Lemma 5.5, obtaining

$$a(y, -\nabla_y \varphi(x, \xi; y)) \underset{\varphi}{\sim} \sum_{|\alpha| \leq |\beta| \leq N} q_{\varphi, \alpha, \beta}(x, \xi; y) \partial_{\xi}^{\alpha} \mathfrak{D}_{\varphi, x}^{\beta} a(x, \xi) + \tilde{R}_N(x, \xi; y),$$

where

$$\tilde{R}_N(x, \xi; y) = \sum_{|\alpha| \leq |\beta| = N+1} q_{\varphi, \alpha, \beta}(x, \xi; y) \int_0^1 \frac{d^\alpha}{d\xi^\alpha} \frac{d^\beta}{dy^\beta} a(y, -\nabla_y \varphi(x, \xi; y)) \Big|_{y=y_t} dt$$

and $q_{\varphi, \alpha, \beta}$ is positively homogeneous of degree $|\alpha| - |\beta|$ in ξ . Applying Corollary 5.12, we obtain that $\tilde{R}(x, \xi; y) \underset{\varphi}{\sim} R(x, \xi; y) \in \mathcal{S}_{\rho, \delta'}^{m - (\rho - \delta'')(N+1)}$. To conclude the claim, we just need to notice that $\delta'' = \delta$ since $1 - \rho \leq 2\delta$ and therefore the order of the remainder decreases to $-\infty$ as N becomes larger.

Finally, in order to obtain (5.12), we just need to remove the dependence on y of the coefficients $q_{\varphi, \alpha, \beta}(x, \xi; y)$ in formula (5.13) by means of the procedure described in Lemma 5.8. \square

Corollary 5.14. *Let $a \in \mathcal{S}_{\rho, \delta}^{m_1}(\varphi, \mathcal{M}, \Omega^{1/2})$, $b \in \mathcal{S}_{\rho, \delta}^{m_2}(\varphi, \mathcal{M}, \Omega^{1/2})$ and $c \in \mathcal{S}^l(T^*\mathcal{M})$. If $1 - \rho \leq 2\delta$, then*

$$c(x, \xi; y) a(x, \xi) b(y, -\nabla_y \varphi(x, \xi; y)) \underset{\varphi}{\sim} \sum_{\alpha, \beta, \gamma} c_{\varphi, \alpha, \beta, \gamma}(x, \xi) \partial_\xi^\alpha a(x, \xi) \partial_\xi^\beta \mathfrak{D}_{\varphi, x}^\gamma b(x, \xi),$$

where $c_{\varphi, \alpha, \beta, \gamma} \in \mathcal{S}^{l + \min\{|\alpha| - |\beta|, 0\}}(T^*\mathcal{M})$. In particular, the coefficient $c_{\varphi, 0, 0, 0}(x, \xi)$ coincides with $c(x, \xi; y) \Big|_{y=x}$.

Proof. It is sufficient to repeat the same procedure used in Proposition 5.13. \square

We are now ready to present the central result of this thesis. The following result puts together all the properties of the φ symbols and amplitudes presented in Chapter 3, the new version of the stationary phase formula proved in Chapter 4, and the material contained so far in this chapter. Carefully combining these results, we are able to study in detail the operator AB^* and derive results for adjoints and compositions, with $A \in L_{\rho, \delta}^{m_1}(\varphi, \mathcal{M}, \Omega^{1/2})$ and $B \in L_{\rho, \delta}^{m_2}(\varphi, \mathcal{M}, \Omega^{1/2})$. The only

element requiring some further analysis is the remainder term. We need to make sure to estimate its order and check that it decreases to $-\infty$ as the number of terms in the series grows. Precisely, we still need to control R^{3N} appearing in Theorem 4.8. This is done once again by extracting all the factors vanishing on the diagonal \mathcal{D} in $\mathcal{M} \times \mathcal{M}$ and turning them into differentiation by ξ . This produces an improvement in the order of the amplitude and allows one to relax the classical condition from $1 - \rho \leq \delta$ to $1 - \rho \leq 2\delta$ or even $\rho > 1/3$.

Theorem 5.15. *Let $A \in L_{\rho,\delta}^{m_1}(\varphi, \mathcal{M}, \Omega^{1/2})$ and $B \in L_{\rho,\delta}^{m_2}(\varphi, \mathcal{M}, \Omega^{1/2})$. If $1 - \rho \leq 2\delta$, then $AB^* \in L_{\rho,\delta}^{m_1+m_2}(\varphi, \mathcal{M}, \Omega^{1/2})$ acting in the space of $1/2$ -densities and*

$$\sigma_{\varphi, AB^*}(x, \xi) \sim \sum_{\alpha, \beta, \beta', \gamma} p_{\varphi, \alpha, \beta, \beta', \gamma}(x, \xi) \partial_{\xi}^{\alpha} \sigma_{\varphi, A}(x, \xi) \partial_{\xi}^{\beta + \beta'} \mathfrak{D}_{\varphi, x}^{\gamma} \overline{\sigma_{\varphi, B}(x, \xi)}, \quad (5.14)$$

where $p_{\varphi, \alpha, \beta, \beta', \gamma} \in \mathcal{S}^{\frac{1}{3}|\beta'| + \min\{|\alpha| + |\beta| - |\gamma|, 0\}}(T^*\mathcal{M})$. Further, $p_{\varphi, 0, 0, 0, 0}(x, \xi) = 1$ and therefore

$$\sigma_{\varphi, AB^*}(x, \xi) \sim \sigma_{\varphi, A}(x, \xi) \overline{\sigma_{\varphi, B}(x, \xi)} \quad \text{mod } \mathcal{S}_{\rho, \delta}^{m_1+m_2-\rho+\max\{\delta, 1/3\}}(\varphi, T^*\mathcal{M}).$$

Remark 5.16. The terms in the right hand side of (5.14) form an asymptotic. In fact, since $\rho > 1/3$ one has

$$\begin{aligned} & \frac{1}{3}|\beta'| + \min\{|\alpha| + |\beta| - |\gamma|, 0\} - \rho(|\alpha| + |\beta| + |\beta'|) + \delta|\gamma| \\ & \leq -(\rho - \delta) \max\{|\alpha| + |\beta|, |\gamma|\} - |\beta'| \left(\rho - \frac{1}{3} \right) \rightarrow -\infty \end{aligned}$$

as $|\alpha| + |\beta| + |\beta'| + |\gamma| \rightarrow \infty$. In particular, Theorem 5.15 can be reformulated assuming only the condition $\rho > 1/3$ on the type of the operators A and B . In this case, $AB^* \in L_{\rho, \delta''}^{m_1+m_2}(\varphi, \mathcal{M}, \Omega^{1/2})$.

In the following proof, one shall assume to work under the same assumptions of Chapter 4. In particular, all the computations are carried out in local coordinates.

Proof. The composition AB^* is an operator, acting in the space of $1/2$ -densities, whose Schwartz kernel is smooth outside the diagonal. Without loss of generality, we assume \mathbf{p} is equal to zero outside some small neighbourhood of the diagonal. Then the Schwartz kernel of the operator AB^* modulo operators with smooth kernel is represented by the oscillatory integral

$$\int e^{i[\varphi(x,\xi;z)-\varphi(y,\eta;z)]} \sigma_{\varphi,A}(x,\xi) \overline{\sigma_{\varphi,B}(y,\eta)} \mathbf{p}(x;z) \overline{\mathbf{p}(y;z)} dz d\eta d\xi. \quad (5.15)$$

We want apply stationary phase formula to

$$\begin{aligned} & \int e^{i[\varphi(x,\xi;z)-\varphi(y,\eta;z)]} \overline{\sigma_{\varphi,B}(y,\eta)} \mathbf{p}(x;z) \overline{\mathbf{p}(y;z)} dz d\eta \\ &= |\xi|^n \mathbf{p}(x;y) \int e^{i|\xi|[\varphi(x,\xi;z)-\varphi(y,\eta;z)]} \frac{\mathbf{p}(x;z) \overline{\mathbf{p}(y;z)}}{\mathbf{p}(x;y)} \overline{\sigma_{\varphi,B}(y, |\xi|\eta)} dz d\eta, \end{aligned}$$

after the change of variables $\xi \rightarrow |\xi|\xi$ and $\eta \rightarrow |\xi|\eta$. Furthermore, we define $p(x;y;z) = \frac{\mathbf{p}(x;z) \overline{\mathbf{p}(y;z)}}{\mathbf{p}(x;y)}$. Notice that because of our assumptions we can identify it with a smooth function.

We observe that

$$|\partial_\eta^\alpha \partial_z^\beta (p(x;y;z) \overline{\sigma_{\varphi,B}(y, |\xi|\eta)})| \leq C \langle \xi \rangle_x^{m_2 + (1-\rho)|\alpha|},$$

Moreover, the only stationary point of the phase function

$$\Phi(\eta; z) = \varphi(x, \xi; z) - \varphi(y, \eta; z),$$

when we regard x, ξ and y as parameters, is $(\eta_0; z_0) = (-\nabla_y \varphi(x, \xi; y); y)$. The Hessian matrix of Φ at $(\eta_0; z_0)$ is

$$H_{\eta, z} \Phi(\eta_0; z_0) = \begin{pmatrix} 0 & I \\ I & H_z \Phi(\eta_0; z_0) \end{pmatrix}.$$

Because of these considerations, we can apply Theorem 4.8 where $|\xi|$ plays the role of the parameter τ . Notice that $\Phi(\eta_0; z_0) = \varphi(x, \xi; y)$ and therefore

$$\begin{aligned} & \int e^{i[\varphi(x, \xi; z) - \varphi(y, \eta; z)]} p(x; y; z) \overline{\sigma_{\varphi, B}(y; |\xi| \eta)} dz d\eta \\ &= e^{i\varphi(x, \xi; y)} \left(\sum_{l=0}^{N-1} |\xi|^{-l} P_l \overline{\sigma_{\varphi, B}(x, \xi; y)} + R^{(3N)}(x, \xi; y) \right) \\ & \qquad \qquad \qquad \text{mod } \mathcal{O}(\langle \xi \rangle_x^{m+2N(1-3\rho')}) \end{aligned}$$

Here, the operator P_l and the remainder $R^{(3N)}$ appearing above are defined as follows. Let

$$\begin{aligned} g_0(\eta; z) &= \varphi(x, \xi; z) - \varphi(y, \eta; z) - \varphi(x, \xi; y) \\ &\quad - \langle z - y, \eta - \eta_0 \rangle - \frac{1}{2} \langle H_z \Phi(\eta_0; y)(z - y), z - y \rangle \end{aligned}$$

and

$$P = -2\langle \nabla_\eta, \nabla_z \rangle + \langle H_z \Phi(\eta_0; y) \nabla_\eta, \nabla_\eta \rangle. \quad (5.16)$$

Define

$$P_l \sigma_{\varphi, B}(x, \xi; y) = \sum_{j-k=l} \sum_{2j \geq 3k} \frac{i^{-l} 2^{-j}}{k! j!} P^j \left(g_0^k(\eta; z) p(x; y; z) \overline{\sigma_{\varphi, B}(x, |\xi| \eta)} \right) \Big|_{z=y; \eta=\eta_0}. \quad (5.17)$$

Furthermore, we have

$$R^{(3N)}(x, \xi; y) = \sum_{k=0}^{2N-1} C_{k,N} |\xi|^{-3N+k} \int \Psi(\tilde{\eta}; \tilde{z}) \\ \times \mathcal{F}_{(\eta;z) \rightarrow (\tilde{\eta};\tilde{y})} (P^{3N}(g_0^k(\eta; z) p(x; y; z) \overline{\sigma_{\varphi,B}(x, |\xi|\eta|)}) d\tilde{z} d\tilde{\eta},$$

$$\text{with } \Psi(\tilde{\eta}; \tilde{z}) = \int_0^1 e^{-\frac{1}{2}it|\xi|^{-1}(2\langle w-y, \xi-\eta_0 \rangle - \langle H_z^{-1}\Phi(\eta_0; y)(\tilde{\eta}-\eta_0), \tilde{\eta}-\eta_0 \rangle)} (1-t)^{N-1} dt.$$

Note that P_l is a differential operator of order $2j$. Differentiation in z is harmless. In order for the corresponding term to be nonzero, $3k$ of derivatives in η must go to g^k . The remaining $2l - k$ derivatives in η may go either to g_0^k or to $\sigma_{\varphi,B}$. The differentiation of g_0^k is harmless, since this function and is bounded in η with all its derivatives. Observe that

$$H_z \Phi(\eta_0; z_0) = H_y \varphi(x, \xi; y) - H_z \varphi(y, \eta_0; z) \Big|_{z=y} = \mathcal{O}(|x - y|). \quad (5.18)$$

It follows from (5.17) and (5.18) that

$$P_l \sigma_{\varphi,B}(x, \xi; y) = \sum_{|\nu| \leq 2l} q_\nu(x, \xi; y) (\partial_\eta^\nu \overline{\sigma_{\varphi,B}})(y, -\nabla_y \varphi(x, \xi; y)),$$

where $q_\nu(x, \xi; y)$ is positively homogeneous of degree 0 in ξ . Moreover, $q_\nu(x, \xi; y)$ has a zero of order $\max\{|\nu| - l, 0\}$ at $y = x$ and $q_0(x, \xi; y) \Big|_{y=x} = 1$ since $\mathbf{p}(x, x) = 1$.

Analogously, one can show that

$$P^{3N}(g_0^k(\eta; z) p(x; y; z) \overline{\sigma_{\varphi,B}(x, |\xi|\eta|)}) = \sum_{|\nu| \leq 6N} r_\nu(x, \xi; y; z) (\partial_\eta^\nu \overline{\sigma_{\varphi,B}})(y, |\xi|\eta|)$$

with $r_\nu(x, \xi; y; z)$ positively homogeneous of degree $-3N + k + |\nu|$ in ξ and has a zero at $y = x$ of order $3k + \max\{|\nu| - 3N, 0\}$. Since $\|\mathcal{F}g\|_{L_1} \leq \|g\|_{H^s}$ with $s > n/2$

and $\Psi(\tilde{y}, \xi)$ is bounded, by Lemma 4.4 we obtain

$$|R^{(3N)}(x, \xi; y)| \leq C \langle \xi \rangle_x^{m-3N(1-3\rho') + n(1-\rho)},$$

with $\rho' = \min\{\rho, 1/2\}$. Observe that $m - 3N(1 - 3\rho') + n(1 - \rho) \rightarrow -\infty$ as $N \rightarrow \infty$ since $\rho > 1/3$. Keeping into account (5.15), we obtain

$$\sigma_{\varphi, AB^*}(x, \xi) \sim \sigma_{\varphi, A}(x, \xi) \sum_{l=0}^{\infty} \sum_{|\nu| \leq 3l} \tilde{p}_{\varphi, \nu}(x, \xi; y) (\partial_{\eta}^{\nu} \overline{\sigma_{\varphi, B}})(y, -\nabla_y \varphi(x, \xi; y)),$$

where $\tilde{p}_{\varphi, \nu}(x, \xi; y) \in \mathcal{S}^{-l + \max\{|\nu| - l, 0\}}(T^*\mathcal{M} \times \mathcal{M})$ and $\tilde{p}_{\varphi, \mathbf{0}}(x, \xi; y) \Big|_{y=x} = 1$.

By using Corollary 5.14 and rearranging the terms of the asymptotic expansion we obtain

$$\sigma_{\varphi, AB^*}(x, \xi) \sim \sum_{\alpha, \beta, \beta', \gamma} p_{\varphi, \alpha, \beta, \beta', \gamma}(x, \xi) \partial_{\xi}^{\alpha} \sigma_{\varphi, A}(x, \xi) \partial_{\xi}^{\beta + \beta'} \mathfrak{D}_{\varphi, x}^{\gamma} \overline{\sigma_{\varphi, B}(x, \xi)},$$

with $p_{\varphi, \alpha, \beta, \beta', \gamma}(x, \xi) \in \mathcal{S}^{|\beta'|/3 + \min\{|\alpha| + |\beta| - |\gamma|, 0\}}(T^*\mathcal{M})$ and $p_{\varphi, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}}(x, \xi) = 1$. \square

Finally, the following two corollaries reformulate the classical results on adjoints and composition of Ψ DOs (Proposition 1.18 and Proposition 1.19) in our setting. These are immediate consequences of Theorem 5.15.

Corollary 5.17. *Let $A \in L_{\rho, \delta}^m(\varphi, \mathcal{M}, \Omega^{1/2})$. If $1 - \rho \leq 2\delta$, then $A^* \in L_{\rho, \delta}^m(\varphi, \mathcal{M}, \Omega^{1/2})$ and*

$$\sigma_{\varphi, A^*}(x, \xi) \sim \sum_{\alpha, \alpha', \beta} p_{\varphi, \alpha, \alpha', \beta}(x, \xi) \partial_{\xi}^{\alpha + \alpha'} \mathfrak{D}_{\varphi, x}^{\beta} \overline{\sigma_{\varphi, A}(x, \xi)},$$

where $p_{\varphi, \alpha, \alpha', \beta}(x, \xi) \in \mathcal{S}^{\frac{1}{3}|\alpha'| + \min\{|\alpha| - |\beta|, 0\}}(T^*\mathcal{M})$. Further, $p_{\varphi, \mathbf{0}, \mathbf{0}, \mathbf{0}}(x, \xi) = 1$ and therefore

$$\sigma_{\varphi, A^*}(x, \xi) \sim \overline{\sigma_{\varphi, A}(x, \xi)} \quad \text{mod } \mathcal{S}_{\rho, \delta}^{m - \rho + \max\{\delta, 1/3\}}(\varphi, T^*\mathcal{M}).$$

Proof. This is an immediate consequence of Theorem 5.15 and Proposition 5.10. \square

Corollary 5.18. *Let $A \in L_{\rho,\delta}^{m_1}(\varphi, \mathcal{M}, \Omega^{1/2})$, $B \in L_{\rho,\delta}^{m_2}(\varphi, \mathcal{M}, \Omega^{1/2})$. If $1 - \rho \leq 2\delta$, then $AB \in L_{\rho,\delta}^{m_1+m_2}(\varphi, \mathcal{M}, \Omega^{1/2})$ and*

$$\sigma_{\varphi,AB}(x, \xi) \sim \sum_{\alpha, \beta, \beta', \gamma} p_{\varphi, \alpha, \beta, \beta', \gamma}(x, \xi) \partial_{\xi}^{\alpha} \sigma_{\varphi,A}(x, \xi) \partial_{\xi}^{\beta+\beta'} \mathfrak{D}_{\varphi, x}^{\gamma} \sigma_{\varphi,B}(x, \xi),$$

where $p_{\varphi, \alpha, \beta, \beta', \gamma}(x, \xi) \in \mathcal{S}^{\frac{1}{3}|\beta'| + \min\{|\alpha| + |\beta| - |\gamma|, 0\}}(T^*\mathcal{M})$. Further, $p_{\varphi, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}}(x, \xi) = 1$ and therefore

$$\sigma_{\varphi,AB}(x, \xi) \sim \sigma_{\varphi,A}(x, \xi) \sigma_{\varphi,B}(x, \xi) \quad \text{mod } \mathcal{S}_{\rho,\delta}^{m_1+m_2-\rho+\max\{\delta, 1/3\}}(\varphi, T^*\mathcal{M}).$$

Proof. This is an immediate consequence of Theorem 5.15 and Corollary 5.17. \square

Remark 5.19. Analogously to Theorem 5.15, one can reformulate the results for the compositions and the adjoints replacing the condition $1 - \rho \leq 2\delta$ with the less restrictive $\rho > 1/3$. The consequence of this less restrictive assumption is that the resulting operators which are slightly less regular. In fact in this case, $A^* \in L_{\rho,\delta''}^m(\varphi, \mathcal{M}, \Omega^{1/2})$ and $AB \in L_{\rho,\delta''}^{m_1+m_2}(\varphi, \mathcal{M}, \Omega^{1/2})$, where we recall $\delta \leq \delta'' = \max\left\{\frac{1-\rho}{2}, \delta\right\}$.

Chapter 6

Applications

In this chapter the φ -symbolic calculus, developed from the notion of global phase function presented in Chapter 3 and Chapter 5, finds its fulfilment. This theory has already the desirable effect of extending the basic properties of symbols and the composition formulae of Ψ DOs assuming just $0 \leq \delta < \rho \leq 1$ and $\rho > 1/3$, without the use of differential geometry. Now we apply the results obtained so far to review some well-known applications and try to extend the class of admissible operators for them. In [16, Sections 9 and 11], Safarov managed to obtain some encouraging results on elliptic Ψ DOs, boundedness of Ψ DOs and functional calculus. We choose to consider the same three applications and revise them in light of our new approach. We obtain positive results which constitute to a certain extent a significant improvement with respect to the previous ones.

In Section 6.1, we discuss the classical result concerning parametrices of an elliptic Ψ DO. After defining the class $HS_{\rho,\delta}^m(\varphi, T^*\mathcal{M})$ of elliptic symbols, we prove that an operator A from the class $L_{\rho,\delta}^m(\varphi, \mathcal{M}, \Omega^{1/2})$ with φ -symbol in the class $HS_{\rho,\delta}^m(\varphi, T^*\mathcal{M})$ has a parametrix $B \in L_{\rho,\delta}^{-m}(\varphi, \mathcal{M}, \Omega^{1/2})$ when $1 - \rho \leq 2\delta$. This

result will be used later to prove the boundedness of the operators $L_{\rho,\delta}^m(\varphi, \mathcal{M}, \Omega^{1/2})$ in Sobolev spaces.

In Section 6.2, we analyse the classical results on the boundedness on a closed manifold \mathcal{M} . In particular, first we show that $L_{\rho,\delta}^0(\varphi, \mathcal{M}, \Omega^{1/2})$ is bounded in $L^2(\mathcal{M}, \Omega^{1/2})$. By means of the classical technique based on the composition formula (see for example [11]), we extend the result to Sobolev spaces. In [16, Section 9], Safarov proved an analogous result for a manifold \mathcal{M} provided with a symmetric linear connection.

In Section 6.3 we deal with functions of an elliptic classical pseudodifferential operator A on a closed manifold. It appears that for some functions ω in some natural subset $S_\rho^m(\mathbb{R})$ of the smooth functions $\mathcal{C}^\infty(\mathbb{R})$, the operator $\omega(A)$ is a Ψ DO under some conditions. In [10, Section 12.1] Taylor introduces this class for developing functional calculus. These results have already been known for $\rho \in (1/2, 1]$. In [16, Section 11], Safarov presents a technique to deal with functions $\omega \in S_\rho^m(\mathbb{R})$ of operators of the form $A_\nu = (\Delta + \nu)^{1/2}$, with ν a first order self-adjoint differential operator on \mathcal{M} . Because the discussion is limited just to this operator, he manages to prove that $\omega(A_\nu)$ is a Ψ DO for $\omega \in S_\rho^m(\mathbb{R})$ for $\rho \in (0, 1]$ when the manifold is provided with a symmetric linear connection. Instead, our method applies to elliptic classical pseudodifferential operators but it requires the usual condition $\rho \in (1/3, 1]$ and the choice of a specific global phase function satisfying an requirement involving the operators $\mathfrak{D}_{\varphi,x}^{e_j}$.

Recall that we consider the case of a closed manifold \mathcal{M} . Therefore, $L^2(\mathcal{M}, \Omega^{1/2}) = L_{\text{comp}}^2(\mathcal{M}, \Omega^{1/2}) = L_{\text{loc}}^2(\mathcal{M}, \Omega^{1/2})$.

6.1 Elliptic Operators

In this section, we prove that a Ψ DO $A \in L_{\rho,\delta}^m(\varphi, \mathcal{M}, \Omega^{1/2})$, whose φ -symbol satisfies suitable conditions, has an inverse $B \in L_{\rho,\delta}^{-m}(\varphi, \mathcal{M}, \Omega^{1/2})$ or $B \in L_{\rho,\delta''}^{-m}(\varphi, \mathcal{M}, \Omega^{1/2})$ when $1 - \rho \leq 2\delta$ or $\rho > 1/3$ respectively. We need to assume some ellipticity or hypoellipticity conditions on the φ -symbol. These are a transposition in our setting of the comparable requirements in the classical theory (see for example [11, Section 5]). We choose to consider only elliptic operators since this suffices for our purposes. However, a simple modification of the arguments below leads to similar results in the case of hypoelliptic operators.

Let $HS_{\rho,\delta}^m(\varphi, T^*\mathcal{M})$ be the subclass of symbols $\sigma \in S_{\rho,\delta}^m(\varphi, T^*\mathcal{M})$ satisfying the following condition: there exists a positive constant $c_{\mathcal{M}}$ such that

$$|\sigma(x, \xi)| \geq c_{\mathcal{M}} \langle \xi \rangle_x^m \quad \forall (x, \xi) \in T^*\mathcal{M} \text{ such that } x \in \mathcal{M} \text{ and } \langle \xi \rangle_x \geq c_{\mathcal{M}}. \quad (6.1)$$

Definition 6.1. We say that a Ψ DO $A \in L_{\rho,\delta}^m(\varphi, \mathcal{M}, \Omega^{1/2})$ is elliptic if its symbol $\sigma_{\varphi,A} \in HS_{\rho,\delta}^m(\varphi, T^*\mathcal{M})$.

Lemma 6.2. Let V_0 be a neighbourhood of the zero section in $T^*\mathcal{M}$ such that for all $(x, \xi) \in V_0$ one has $\langle \xi \rangle_x \leq M/2$ for some positive constant M . Define $\chi \in \mathcal{C}^\infty(T^*\mathcal{M})$ such that

$$\chi(x, \xi) = \begin{cases} 1 & \langle \xi \rangle_x \geq M, \\ 0 & (x, \xi) \in V_0, \end{cases} \quad (6.2)$$

If $\sigma \in HS_{\rho,\delta}^m(\varphi, T^*\mathcal{M})$, then $\chi\sigma^{-1} \in S_{\rho,\delta}^{-m}(\varphi, T^*\mathcal{M})$.

Proof. It suffices consider estimates of σ^{-1} and its derivatives for $\langle \xi \rangle_x \geq M$.

First, we observe that $|1/\sigma(x, \xi)| \leq C\langle \xi \rangle_x^{-m}$, for some positive constant C . By induction on $|\beta|$, one has

$$\mathfrak{D}_{\varphi, x}^\beta \sigma^{-1}(x, \xi) = \frac{1}{\sigma^{1+|\beta|}(x, \xi)} \sum_{\mu^{(1)} + \dots + \mu^{(j)} = \beta} \mathfrak{D}_{\varphi, x}^{\mu^{(1)}} \sigma(x, \xi) \cdots \mathfrak{D}_{\varphi, x}^{\mu^{(j)}} \sigma(x, \xi).$$

It follows that $|\mathfrak{D}_{\varphi, x}^\beta [\chi(x, \xi)/\sigma(x, \xi)]| \leq C_{\varphi, \beta} \langle \xi \rangle_x^{-m+\delta|\beta|}$ for some positive constants $C_{\varphi, \beta}$. Analogously, we have

$$\partial_\xi^\alpha \sigma^{-1}(x, \xi) = \frac{1}{\sigma^{1+|\alpha|}(x, \xi)} \sum_{\nu^{(1)} + \dots + \nu^{(k)} = \alpha} \partial_\xi^{\nu^{(1)}} \sigma(x, \xi) \cdots \partial_\xi^{\nu^{(k)}} \sigma(x, \xi).$$

It follows that $|\partial_\xi^\alpha [\chi(x, \xi)/\sigma(x, \xi)]| \leq C_{\varphi, \alpha} \langle \xi \rangle_x^{-m-\rho|\alpha|}$ for some positive constants $C_{\varphi, \alpha}$. This concludes the proof. \square

We recall that a parametrix for an operator A is an operator B such that $AB = BA = I$ modulo an operator with smooth kernel.

Theorem 6.3. *Let $1 - \rho \leq 2\delta$. If $A \in L_{\rho, \delta}^m(\varphi, \mathcal{M}, \Omega^{1/2})$ is elliptic, then it has a parametrix $B \in L_{\rho, \delta}^{-m}(\varphi, \mathcal{M}, \Omega^{1/2})$.*

The following proof is a modification of [16, Theorem 10.4].

Proof. Let $b \in \mathcal{S}_{\rho, \delta}^0(\varphi, T^*\mathcal{M})$ and $\tilde{B} \in L_{\rho, \delta}^{-m}(\varphi, \mathcal{M}, \Omega^{1/2})$ with φ -symbol

$$\sigma_{\varphi, \tilde{B}}(x, \xi) = \frac{\chi(x, \xi)}{\sigma_{\varphi, A}(x, \xi)} b(x, \xi),$$

where χ is as in (6.2). Notice that $\sigma_{\varphi, \tilde{B}} \in \mathcal{S}_{\rho, \delta}^{-m}(\varphi, T^*\mathcal{M})$ by Lemma 6.2. Corollary implies immediately that

$$\sigma_{A\tilde{B}} = b(x, \xi) - r_0(x, \xi) \quad \text{and} \quad \sigma_{\tilde{B}A} = b(x, \xi) - r'_0(x, \xi), \quad (6.3)$$

with $r_0, r'_0 \in \mathcal{S}_{\rho, \delta}^{-\rho + \max\{\delta, 1/3\}}(\varphi, T^*\mathcal{M})$. Let $b_0 = \sigma_{\varphi, I}$ and $b_{j+1} = r_j$, $j = 0, 1, 2, \dots$, where r_j is the symbol which appears in the first equality (6.3) when we replace b by b_j . Then $b_j \in \mathcal{S}_{\rho, \delta}^{-j(\rho + \max\{\delta, 1/3\})}(\varphi, T^*\mathcal{M})$ and we define $B \in L_{\rho, \delta}^{-m}(\varphi, \mathcal{M}, \Omega^{1/2})$ with

$$\sigma_{\varphi, B}(x, \xi) \sim \frac{\chi(x, \xi)}{\sigma_{\varphi, A}(x, \xi)} \sum_{j=0}^{\infty} b_j(x, \xi).$$

One obtains immediately that $AB = I$ modulo $L^{-\infty}(\varphi, \mathcal{M}, \Omega^{1/2})$. By analogy (using the second equality in (6.3)) we construct $B' \in L_{\rho, \delta}^{-m}(\varphi, \mathcal{M}, \Omega^{1/2})$ such that $B'A = I$ modulo $L^{-\infty}(\varphi, \mathcal{M}, \Omega^{1/2})$. Since $B' = B'AB = B$ modulo $L^{-\infty}(\varphi, \mathcal{M}, \Omega^{1/2})$, we also have $BA = I$ modulo $L^{-\infty}(\varphi, \mathcal{M}, \Omega^{1/2})$. \square

Remark 6.4. Using the same technique presented many times in Chapter 5, we can relax the requirement on the type of the Ψ DO A to $\rho > 1/3$. In this case, we can only say that the parametrix B is in the class $B \in L_{\rho, \delta''}^{-m}(\varphi, \mathcal{M}, \Omega^{1/2})$, with $\delta'' = \max\left\{\delta, \frac{1-\rho}{2}\right\}$.

6.2 Boundedness

In this section, we deal with the classical question on the boundedness of Ψ DOs on Sobolev spaces. We choose to follow the classical approach to this problem presented in [11, Section 6]. The structure of this section proceeds as follows. First, we establish the boundedness in $L^2(\mathcal{M})$ for Ψ DOs from the class $L_{\rho, \delta}^0(\varphi, \mathcal{M}, \Omega^{1/2})$. Then we extend the results to the more general context of Sobolev spaces, characterised by means of the Bessel potential.

First, we want to prove that a Ψ DO of order zero extends to a bounded operator from the space of $1/2$ -densities in L^2 into itself.

Theorem 6.5. *Let $1 - \rho \leq 2\delta$ and $A \in L_{\rho,\delta}^0(\varphi, \mathcal{M}, \Omega^{1/2})$, then A is a bounded operator from $L^2(\mathcal{M}, \Omega^{1/2})$ to $L^2(\mathcal{M}, \Omega^{1/2})$.*

We need the following auxiliary lemma for the proof of Theorem 6.5. The proof is adapted from [10, Section 2.6 Lemma 6.2].

Lemma 6.6. *Let $A \in L_{\rho,\delta}^0(\varphi, \mathcal{M}, \Omega^{1/2})$, with $1 - \rho \leq 2\delta$, and $\sigma_{\varphi,A} \in \mathcal{S}_{\rho,\delta}^0(\varphi, \mathcal{M})$ be its φ -symbol. If $\operatorname{Re} \sigma_{\varphi,A}(x, \xi) \geq C > 0$, then there exists a $B \in L_{\rho,\delta}^0(\varphi, \mathcal{M}, \Omega^{1/2})$ such that*

$$\operatorname{Re} A - B^*B \in L^{-\infty}(\mathcal{M}, \Omega^{1/2}),$$

where $\operatorname{Re} A = (A + A^*)/2$.

Proof. We shall construct the φ -symbol of the operator B as an asymptotic expansion $\sigma_{\varphi,B} \sim_{\varphi} \sum \sigma_{\varphi,B_j}$ with $\sigma_{\varphi,B_j} \in \mathcal{S}_{\rho,\delta}^{-j(\rho+\max\{1/3,\delta\})}(\varphi, T^*\mathcal{M})$. First, let

$$\sigma_{\varphi,B_0}(x, \xi) = \sqrt{\operatorname{Re} \sigma_{\varphi,A}(x, \xi)}.$$

Since $\sigma_{\varphi,A} \in \mathcal{S}_{\rho,\delta}^0(\varphi, \mathcal{M})$, it follows that $\sigma_{\varphi,B_0}(x, \xi) \in \mathcal{S}_{\rho,\delta}^0(\varphi, \mathcal{M})$. Furthermore, by Corollary 5.17 Corollary and 5.18, we have

$$\operatorname{Re} A - B_0^*B_0 = R_1 \in L_{\rho,\delta}^{-\rho+\max\{1/3,\delta\}}(\varphi, \mathcal{M}, \Omega^{1/2}).$$

Proceeding by induction, suppose we have terms $\sigma_{\varphi,B_0}, \dots, \sigma_{\varphi,B_{j-1}}$ in the asymptotic expansion. We need $B_j \in L_{\rho,\delta}^{-j(\rho+\max\{1/3,\delta\})}(\varphi, T^*\mathcal{M})(\varphi, \mathcal{M}, \Omega^{1/2})$ such that

$$\begin{aligned} \operatorname{Re} A - ((B_0^* + \dots + B_{j-1}^*) + B_j^*)((B_0 + \dots + B_{j-1}) + B_j) \\ = R_{j+1} \in L_{\rho,\delta}^{-(j+1)(\rho+\max\{1/3,\delta\})}(\varphi, \mathcal{M}, \Omega^{1/2}). \end{aligned} \quad (6.4)$$

By (6.4), one has

$$\begin{aligned}
\operatorname{Re} A - ((B_0^* + \dots + B_{j-1}^*) + B_j^*)((B_0 + \dots + B_{j-1}) + B_j) \\
= R_j + B_j^*(B_0 + \dots + B_{j-1}) + (B_0^* + \dots + B_{j-1}^*)B_j \\
= R_j + B_j^*B_0 + B_0^*B_j, \\
\operatorname{mod} \mathcal{S}_{\rho,\delta}^{-(j+1)(\rho+\max\{1/3,\delta\})}(\varphi, T^*\mathcal{M})
\end{aligned}$$

where $R_j \in \mathcal{S}_{\rho,\delta}^{-j(\rho+\max\{1/3,\delta\})}(\varphi, T^*\mathcal{M})$ is the analogous remainder term in the previous stage. Note that $R_j = R_j^*$ so its principal symbol is real. We require of the φ -symbol of B_j that

$$B_j^*B_0 + B_0B_j = -R_j.$$

Consequently, we pick $B_j = -(1/2)B_0^{-1}R_j$. This concludes the proof of the lemma. \square

Proof (Theorem 6.5). Since $A \in L_{\rho,\delta}^0(\varphi, \mathcal{M}, \Omega^{1/2})$, then $\sigma_{\varphi,A} \in \mathcal{S}_{\rho,\delta}^0(\varphi, \mathcal{M}, \Omega^{1/2})$ and there exists

$$|\sigma_{\varphi,A}(x, \xi)| < M, \quad \text{for all } (x, \xi) \in T^*\mathcal{M}, \quad (6.5)$$

for some positive constant M . It follows that the operator $C = M^2 - A^*A$ has principal symbol $c(x, \xi) = M^2 - |\sigma_{\varphi,A}(x, \xi)|^2 > 0$. So by Lemma 6.6 there is a $B \in \mathcal{S}_{\rho,\delta}^0(\varphi, T^*\mathcal{M})$ such that

$$C - B^*B = M^2 - A^*A - B^*B = -R \in L^{-\infty}(\mathcal{M}, \Omega^{1/2}).$$

Thus

$$\begin{aligned}
\|Au\|_{L^2}^2 &= \langle Au, Au \rangle_{L^2} + \langle Bu, Bu \rangle_{L^2} \\
&\leq M^2\|u\|_{L^2}^2 + \langle Ru, u \rangle.
\end{aligned}$$

This completes the proof of the theorem. \square

Now, we generalise this result to the more general setting of Sobolev spaces. First, we define them. A natural way of defining Sobolev spaces in local coordinates is through the Bessel potential Λ_s .

Let $\Lambda_s \in L^s_{1,0}(\mathcal{M}, \Omega^{1/2})$, $s \in \mathbb{R}$, be an elliptic and such that its symbol $\sigma_{\Lambda_s}(x, \xi) > 0$ for $\xi \neq 0$. For example on a compact Riemannian manifold \mathcal{M} without boundary, with the inverse metric tensor denoted by $\{g^{jk}\}$, one can pick $\sigma(x, \xi) = \left(\sum_{j,k} g^{jk}(x) \xi_j \xi_k\right)^{s/2}$. By Theorem 6.3, we can assume without loss of generality that Λ_{-s} is a parametrix Λ_s for arbitrary $s \in \mathbb{R}$, i.e.

$$\Lambda_{-s} \circ \Lambda_s = I + R_s, \quad (6.6)$$

where R_s is an operator with smooth kernel.

For the moment, we assume $u \in H^s(\mathcal{M}, \Omega^{1/2})$, if $u \in \mathcal{D}'(\mathcal{M}, \Omega^{1/2})$ and $\Lambda_s u \in L^2(\mathcal{M}, \Omega^{1/2})$. We will observe later that these spaces do not depend on the choice of the operator Λ_s .

Theorem 6.7. *Let $1 - \rho \leq 2\delta$ and $A \in L^m_{\rho,\delta}(\varphi, \mathcal{M}, \Omega^{1/2})$, then A is a bounded linear operator from $H^s(\mathcal{M}, \Omega^{1/2})$ to $H^{s-m}(\mathcal{M}, \Omega^{1/2})$, for all $s \in \mathbb{R}$.*

Proof. If $u \in H^s(\mathcal{M}, \Omega^{1/2})$ then, setting $\Lambda_s u = u_0$, we obtain from (6.6) that $u = \Lambda_{-s} u_0 + v$, where $u_0 \in L^2(\mathcal{M}, \Omega^{1/2})$ and $v \in \mathcal{C}^\infty(\mathcal{M}, \Omega^{1/2})$. Therefore

$$\Lambda_{s-m} A u = \Lambda_{s-m} A (\Lambda_{-s} u_0 + v) = \Lambda_{s-m} A \Lambda_{-s} u_0 + \Lambda_{s-m} A v.$$

By Corollary 5.18, $\Lambda_{s-m} A \Lambda_{-s} \in L^0_{\rho,\delta}(\varphi, \mathcal{M}, \Omega^{1/2})$. Consequently, by Theorem 6.5, we have that $\Lambda_{s-m} A \Lambda_{-s} u_0 \in L^2(\mathcal{M}, \Omega^{1/2})$. Moreover, since $v \in \mathcal{C}^\infty(\mathcal{M}, \Omega^{1/2})$, it follows that $\Lambda_{s-m} A v \in \mathcal{C}^\infty(\mathcal{M}, \Omega^{1/2})$. Thus $\Lambda_{s-m} A u \in L^2(\mathcal{M}, \Omega^{1/2})$ or equivalently $A u \in H^{s-m}(\mathcal{M}, \Omega^{1/2})$. \square

We are now in the position to give a definition of Sobolev spaces, not depending on the choice of Λ_s .

Definition 6.8. We write $u \in H^s(\mathcal{M}, \Omega^{1/2})$, if $u \in \mathcal{D}'(\mathcal{M}, \Omega^{1/2})$ and $Au \in L^2(\mathcal{M}, \Omega^{1/2})$ for any $A \in L^s_{\rho, \delta}(\varphi, \mathcal{M}, \Omega^{1/2})$.

The equivalence of Definition 6.8 with the definition of Sobolev spaces given above follows from Theorem 6.7.

Remark 6.9. It is well-known that in local coordinates the Schwartz kernel of $A : H^s(\mathcal{M}, \Omega^{1/2}) \rightarrow H^{s-m}(\mathcal{M}, \Omega^{1/2})$ can be represented by an oscillatory integral with the standard phase function $(x-y) \cdot \theta$ and an amplitude from $S^m_{\rho, \delta}(X \times X \times \mathbb{R}^n)$ with

$$\rho > 1/2 \quad \wedge \quad \delta < \rho \quad (6.7)$$

Therefore Theorem 6.7 gives a new result only when

$$1/3 < \rho \leq 1/2 \quad \wedge \quad \delta < \rho, \quad (6.8)$$

(see Figure 6.1).

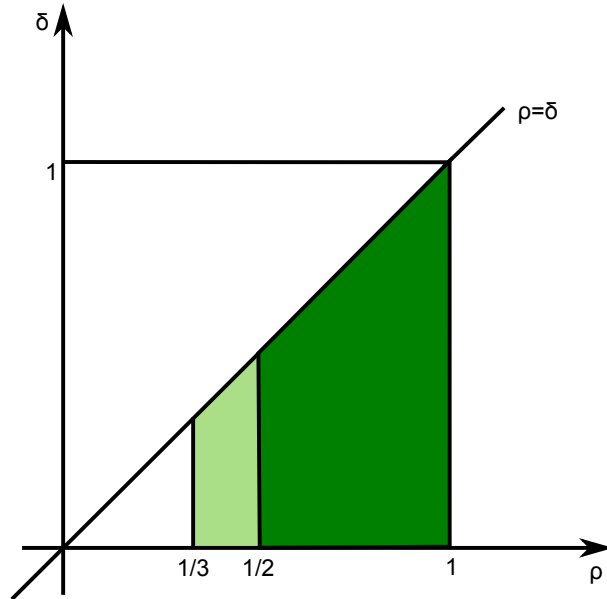


FIGURE 6.1: Condition (6.7) (dark green) and condition (6.8) (light green).

6.3 Functional Calculus

In this section, we deal with functions of a positive self-adjoint elliptic classical pseudodifferential operator A of order m on a closed manifold \mathcal{M} . We introduce the topic of this section recalling some well-established results on the topic. We refer the interested reader to [10, Chapter 12].

If A is self-adjoint and ω is real-valued a Borel measurable function on \mathbb{R} , one can define the operator $\omega(A)$ by means of the spectral theorem (for example see [25]).

In particular, if $\omega(\theta) \in S_{\rho,0}^m(\mathbb{R})$ and $P \in L^1(\mathcal{M})$ is positive, self-adjoint and elliptic with principal symbol p_1 , Taylor proves in [10] that $\omega(P) \in L_{\rho,1-\rho}^m(\mathcal{M})$ provided

$$\rho \in (1/2, 1] \tag{6.9}$$

and its principal symbol is $\omega(p_1)$. These results rely heavily on the composition formula for Ψ DOs on \mathbb{R}^n . The importance lies in the fact that one wants to study fine properties of the operators $\omega(P)$. For instance, we immediately obtain that the operator $\omega(P)$ is pseudolocal.

The symbolic calculus introduced by Safarov in [16] allows to develop functional calculus, understood as above, for the case of operators of the kind

$$P_\nu = (-\Delta + \nu)^{1/2},$$

where Δ is the Laplace operator and ν a first order positive self-adjoint classical Ψ DO on a closed Riemannian manifold. In this setting, Safarov manages to extend the classical theory of functional calculus to the case $\omega(\theta) \in S_{\rho,0}^m(\mathbb{R})$ with

$$\rho \in (0, 1]. \tag{6.10}$$

We restrict to considering the case when the operator A is of order 1. We deal only with first order operators since it turns out that for $P \in L^m(\varphi, \mathcal{M}, \Omega^{1/2})$, one has $P^{1/m} \in L^1(\varphi, \mathcal{M}, \Omega^{1/2})$. Then the principal symbol $a_1(x, \xi)$ of A is real valued and positively homogeneous of degree 1 in ξ . We require that the following two conditions hold

$$a_1(x, \xi) > 0 \quad \text{for } \xi \neq 0, \quad (6.11)$$

$$A > \delta I, \quad \text{for some } \delta > 0. \quad (6.12)$$

Notice that this operator is elliptic and strictly positive. Moreover, condition (6.12) can always be achieved by adding a suitable positive constant.

These assumptions allow us to consider powers A^λ , $\lambda \in \mathbb{R}$, of the operator A and exponential functions of the operator A of the form e^{-itA^λ} . Chapter 10 and 11 of [11] discuss in detail the operators A^λ . We just recall that they are well defined and the operators of the form A^λ are classical pseudodifferential operators of order λ with principal symbol equal to $a_1^\lambda(x, \xi)$. Instead, the operators of the kind e^{itA^λ} are analysed in [16, Section 11].

In this section, we prove that the operator $\omega(A) \in L_{\rho, \frac{1-\rho}{2}}^m(\varphi, \mathcal{M}, \Omega^{1/2})$, with $\omega(\theta) \in S_{\rho, 0}^m(\mathbb{R})$ and

$$\rho \in (1/3, 1]. \quad (6.13)$$

and its symbol coincides with $\omega(a_1(x, \xi))$ modulo terms of lower order. We observe that that condition (6.13) is stronger than (6.10), but it is still weaker than (6.9). However, while Safarov deals with functions of the Laplacian, we extend the functional calculus theory to the far more general class of elliptic classical pseudodifferential operators.

The choice of a suitable phase function φ for the integral representation is fundamental for the proof. Namely, we need to require that the ‘curved gradient’ $\nabla_{\varphi,x}$ defined in (3.8) of the principal φ -symbol of A vanishes on $T^*\mathcal{M}$. Recall that we have freedom in the choice of the global phase function a classical Ψ DO $(1,0)$. From now on we assume that φ is a fixed global phase function such that

$$\nabla_{\varphi,x}a_1(x,\xi) = \nabla_x a_1(x,\xi) - H_y \varphi(x,\xi;y) \Big|_{y=x} \nabla_\xi a_1(x,\xi) = 0. \quad (6.14)$$

It is simple to observe that such global phase function φ exists. In fact, the equation in (6.14) is formally satisfied when

$$H_y \varphi(x,\xi;y) \Big|_{y=x} = \begin{pmatrix} \frac{\partial_{x_1} a_1(x,\xi)}{\partial_{\xi_1} a_1(x,\xi)} & 0 & \cdots & 0 \\ 0 & \frac{\partial_{x_2} a_1(x,\xi)}{\partial_{\xi_2} a_1(x,\xi)} & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & \frac{\partial_{x_n} a_1(x,\xi)}{\partial_{\xi_n} a_1(x,\xi)} \end{pmatrix}.$$

Observe that (6.11) implies that $\partial_{\xi_j} a_1(x,\xi) \neq 0$ for $\xi \neq 0$, $j = 1, \dots, n$. As the singularity at $\xi = 0$ can be removed introducing a suitable cut-off function, by Taylor’s formula, keeping into account that $\partial_{x_j} a_1(x,\xi)/\partial_{\xi_j} a_1(x,\xi)$ is positively homogeneous in ξ of degree 1, we obtain immediately the existence of such a global phase function φ .

Definition 6.10. If $\rho \in (0,1]$, define $S_\rho^m(\mathbb{R})$ the class of functions $\omega \in \mathcal{C}^\infty(\mathbb{R})$ such that

$$|\partial_\theta^j \omega(\theta)| \leq C_j \langle \theta \rangle^{m-j\rho}.$$

Before dealing with operators of the kind $\omega(A)$, where $\omega \in S_\rho^m(\mathbb{R})$, we need to introduce some technical results. In fact, we prove that $\omega(A)$ is a Ψ DO by means

of the Fourier inversion formula. This technique will require the use of operators of the kind $U_\lambda(t) = e^{itA^\lambda}$. The following proposition is devoted to the study of such operators. His proof is based on the same technique used to prove [16, Proposition 11.6].

Proposition 6.11. *Let $\lambda \in [0, 2/3)$ and $U_\lambda(t) = e^{itA^\lambda}$. Then for all $t \in \mathbb{R}$,*

$$U_\lambda(t) \in L_{1-\lambda, \frac{\lambda}{2}}^0(\varphi, \mathcal{M}, \Omega^{1/2}) \quad \text{and} \quad \sigma_{\varphi, U_\lambda(t)}(x, \xi) \underset{\varphi}{\sim} e^{ita_1^\lambda(x, \xi)} b^{(\lambda)}(t; x, \xi),$$

where $b^{(\lambda)} \in \mathcal{C}^\infty(\mathbb{R} \times T^*\mathcal{M})$ and $\partial_t^k b^{(\lambda)} \in \mathcal{S}^0(T^*\mathcal{M})$ for all $k \in \mathbb{N}_0$ and each t fixed.

Moreover,

$$b^{(\lambda)}(t; x, \xi) \sim \sigma_{\varphi, I}(x, \xi) + \sum_{j=1}^{\infty} (it)^j b_j^{(\lambda)}(x, \xi). \quad (6.15)$$

where $b_j^{(\lambda)} \in S^{j(\lambda-2/3)}(T^*\mathcal{M})$, for all $j \in \mathbb{N}$.

Remark 6.12. We observe that for $\tilde{\rho} = 1 - \lambda$, the operator $U_\lambda(t)$ is an element of the class $L_{\tilde{\rho}, \frac{1-\tilde{\rho}}{2}}^0(\varphi, \mathcal{M}, \Omega^{1/2})$. In particular, the type of the operator consists of the least regular couple satisfying the condition $1 - \rho \leq 2\delta$ under which the composition formula holds.

The proof of Proposition 6.11 relies on three lemmas, which we now present.

The first lemma gives us an insight on the basic properties of the φ -symbol $\omega(a_1)$.

Since

$$|\partial_{x_j} \omega(a_1(x, \xi))| \leq C \langle \xi \rangle_x^{m+1-\rho},$$

one would expect the type of the $\omega(a_1)$ to be $(\rho, 1 - \rho)$ and therefore the condition $\rho > 1/2$. However, because of the condition $\nabla_{\varphi, x} a_1 = 0$ on the global phase function, we have an improvement in the type of the symbol.

Lemma 6.13. *Let $\rho \in (1/3, 1]$ and $\omega \in S_\rho^0(\mathbb{R})$. Then $\omega(a_1(x, \xi)) \in \mathcal{S}_{\rho, \frac{1-\rho}{2}}^0(\varphi, T^*\mathcal{M})$.*

Proof. We notice that

$$\mathfrak{D}_{\varphi,x}^\beta \omega(a_1(x, \xi)) = \sum_{\substack{k \leq |\beta| \\ \beta^{(1)} + \dots + \beta^{(k)} = \beta}} \omega^{(k)}(a_1(x, \xi)) \mathfrak{D}_{\varphi,x}^{\beta^{(1)}} a_1(x, \xi) \cdots \mathfrak{D}_{\varphi,x}^{\beta^{(k)}} a_1(x, \xi), \quad (6.16)$$

$$\omega^{(k)}(\theta) = \partial_\theta^k \omega(\theta).$$

We obtain immediately that $|\mathfrak{D}_{\varphi,x}^\beta a_1(x, \xi)|$ is estimated by $C\langle \xi \rangle_x^{m+\frac{1-\rho}{2}|\beta|}$. In fact, we assumed that $\nabla_{\varphi,x} a_1(x, \xi) = 0$, $1 \leq j \leq n$, and therefore we have that $|\beta^{(j)}| \geq 2$ which implies $k \leq |\beta|/2$.

Finally we observe that differentiation in ξ improves the order by ρ . In fact, by the chain rule

$$\partial_\xi^\alpha \omega^{(k)}(a_1(x, \xi)) = \sum_{\substack{j \leq |\alpha| \\ \alpha^{(1)} + \dots + \alpha^{(j)} = \alpha}} \omega^{(k+j)}(a_1(x, \xi)) \partial_\xi^{\alpha^{(1)}} a_1(x, \xi) \cdots \partial_\xi^{\alpha^{(j)}} a_1(x, \xi).$$

Therefore, in the worst case scenario, one has $|\partial_\xi^\alpha \omega^{(k)}(a_1(x, \xi))| \leq C_\alpha \langle \xi \rangle_x^{m-\rho(|\alpha|+k)}$. Keeping into account this and the fact that $\partial_\xi^\nu \mathfrak{D}_{\varphi,x}^\mu a_1(x, \xi)$ is positively homogeneous of degree $1 - |\nu|$ in ξ , we just need to use the Leibnitz rule on the right side of (6.16) to obtain

$$|\partial_\xi^\alpha \mathfrak{D}_{\varphi,x}^\beta a_1(x, \xi)| \leq C \langle \xi \rangle_x^{m-\rho|\alpha|+\frac{1-\rho}{2}|\beta|}.$$

□

The second lemma gives us additional information concerning $\omega(a_1)$ and Ψ DOs associated with it. Precisely, we investigate the composition of such Ψ DOs with a classical Ψ DO.

Lemma 6.14. *Let $P \in L^{m_1}(\mathcal{M}, \Omega^{1/2})$, $\omega \in S_\rho^0(\mathbb{R})$, $\rho \in (1/3, 1]$ and Q be a Ψ DO with a φ -symbol $\sigma_{\varphi,Q}(x, \xi) = \omega(a_1(x, \xi))q(x, \xi)$, where $q \in \mathcal{S}^{m_2}(T^*\mathcal{M})$. Then the*

φ -symbol of the composition PQ is

$$\sigma_{\varphi,PQ}(x, \xi) \sim \sum_{k=0}^{\infty} \omega^{(k)}(a_1(x, \xi)) q_k(x, \xi), \quad (6.17)$$

where $q_k \in \mathcal{S}^{m_1+m_2+k/3}$ for $k \geq 0$ and

$$q_0(x, \xi) = \sigma_{\varphi,P}(x, \xi) q(x, \xi) \mod \mathcal{S}^{m_1+m_2-2/3}(T^*\mathcal{M}).$$

Proof. First, by Lemma 6.13 it follows that $Q \in L_{\rho, \frac{1-\rho}{2}}^{m_2}(\varphi, \mathcal{M}, \Omega^{1/2})$. By the composition formula (Corollary 5.18), $PQ \in L_{\rho, \frac{1-\rho}{2}}^{m_1+m_2}(\varphi, \mathcal{M}, \Omega^{1/2})$ and

$$\sigma_{\varphi,PQ}(x, \xi) \sim \sum_{\alpha, \beta, \beta', \gamma} p_{\varphi, \alpha, \beta, \beta', \gamma}(x, \xi) \partial_{\xi}^{\alpha} \sigma_{\varphi,P}(x, \xi) \partial_{\xi}^{\beta+\beta'} \mathfrak{D}_{\varphi, x}^{\gamma} \sigma_{\varphi,Q}(x, \xi), \quad (6.18)$$

where $p_{\varphi, \alpha, \alpha', \beta, \gamma}(x, \xi) \in \mathcal{S}^{|\beta'|/3 + \min\{|\alpha|+|\beta|-|\gamma|, 0\}}(T^*\mathcal{M})$. Rearranging the terms in the asymptotic expansion (6.18) according to the order of differentiation of ω , we obtain an asymptotic expansion of the form (6.17). Notice that

$$q_0(x, \xi) \sim \sum_{\alpha, \beta, \beta', \gamma} p_{\varphi, \alpha, \beta, \beta', \gamma}(x, \xi) \partial_{\xi}^{\alpha} \sigma_{\varphi,P}(x, \xi) \partial_{\xi}^{\beta+\beta'} \mathfrak{D}^{\gamma} q(x, \xi) \in \mathcal{S}^{m_2}(T^*\mathcal{M}).$$

Since $p_{\varphi, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}} = 1$,

$$q_0(x, \xi) = \sigma_{\varphi,P}(x, \xi) q(x, \xi) \mod \mathcal{S}^{m_1+m_2-2/3}(T^*\mathcal{M}).$$

In general, the worst case scenario for the coefficient q_k of the k -th term of the series is when $|\beta'| = k$, then $p_{\varphi, \alpha, \beta, \beta', \gamma}(x, \xi) \in \mathcal{S}^{k/3}(T^*\mathcal{M})$ and consequently $q_k \in \mathcal{S}^{m_1+m_2+k/3}(T^*\mathcal{M})$. \square

We are now ready to prove the result concerning the operator $U_{\lambda}(t) = e^{itA^{\lambda}}$,

$\lambda \in [0, 2/3)$. We will use some well-known facts concerning self-adjoint operators. For full details on the topic, we refer the interested reader to [26, Section 7.6]. We just recall that the operator $U_\lambda(t)$ is a unique solution to the Cauchy problem

$$\begin{cases} D_t U_\lambda(t) - A^\lambda U_\lambda(t) = 0, \\ U_\lambda(0) = I. \end{cases} \quad (6.19)$$

In addition, one has

$$e^{itA^\lambda} A^\lambda = A^\lambda e^{itA^\lambda} \quad \text{and} \quad U_\lambda^*(t) = U_\lambda^{-1}(t) = U_\lambda(-t) = e^{-itA^\lambda}$$

Proof (Proposition 6.11). First, we construct a Ψ DO $V_\lambda(t) \in L_{1-\lambda, \frac{\lambda}{2}}^0(\varphi, \mathcal{M}, \Omega^{1/2})$ smoothly dependent on t which is a solution to the Cauchy problem

$$\begin{cases} D_t V_\lambda(t) - A^\lambda V_\lambda(t) = K(t), \\ V(0) = I, \end{cases}$$

where $K(t)$ has smooth kernel with smooth dependence on t . We will then show that that $U_\lambda(t) = V_\lambda(t)$ modulo an operator with smooth kernel smoothly dependent on t .

The function $f_t(\theta) = e^{it\theta^\lambda}$ belongs to $S_{1-\lambda}^0(\mathbb{R})$ (outside a neighbourhood of $|\theta| = 0$). Let $B(t)$ be a Ψ DO with a symbol of the form $e^{ita_1^\lambda(x, \xi)} b(t; x, \xi)$, where $b \in \mathcal{S}^0(\mathbb{R} \times T^*\mathcal{M})$. Then by Lemma 6.14,

$$\sigma_{\varphi, A^\lambda B(t)}(t; x, \xi) \sim e^{ita_1^\lambda(x, \xi)} a_1^\lambda(x, \xi) b(t; x, \xi) + e^{ita_1^\lambda(x, \xi)} \mathcal{L}_\lambda b(t; x, \xi),$$

where $\mathcal{L}_\lambda b(t; x, \xi) \in \mathcal{S}^{\lambda-2/3}(T^*\mathcal{M})$ for all $t \in \mathbb{R}$. Moreover, the explicit formula

(6.18) implies that the operator \mathcal{L}_λ does not involve differentiation in t or multiplication by functions of t . It follows that we can rearrange term of $\mathcal{L}_\lambda b(t; x, \xi)$ and obtain an asymptotic expansion of the form

$$\mathcal{L}_\lambda b(t; x, \xi) \sim \sum_{k=0}^{\infty} t^k \mathcal{L}_\lambda^{(k)} b(t; x, \xi), \quad (6.20)$$

where $\mathcal{L}_\lambda^{(k)} b(t; x, \xi) \in \mathcal{S}^{(k+1)(\lambda-2/3)}$ and

$$t^j \partial_t^l (\mathcal{L}_\lambda b(t; x, \xi)) = \mathcal{L}_\lambda (t^j \partial_t^l b(t; x, \xi)) \quad \text{for all } t \in \mathbb{R} \text{ and } j, k, l \in \mathbb{N}_0. \quad (6.21)$$

Let $V_\lambda(t)$ be a Ψ DO such that $\sigma_{V_\lambda(t)} = e^{ita_1^\lambda(x, \xi)} b^{(\lambda)}(t; x, \xi)$ (outside a neighbourhood of $|\xi| = 0$), where

$$b^{(\lambda)} \sim \sum_{k=0}^{\infty} \tilde{b}_k^{(\lambda)} \quad \text{with} \quad \tilde{b}_0^{(\lambda)}(t; x, \xi) = \sigma_{I, \varphi}(x, \xi)$$

$$\text{and} \quad \tilde{b}_k^{(\lambda)}(t; x, \xi) = i \int_0^t \mathcal{L}_\lambda \tilde{b}_{k-1}^{(\lambda)}(s; x, \xi) ds, \quad k \in \mathbb{N}. \quad (6.22)$$

Then $b^{(\lambda)}(0; x, \xi) = \sigma_{I, \varphi}(x, \xi)$ and so $V_\lambda(0) = I$. The operator $(D_t - A^\lambda)V_\lambda(t)$ is a Ψ DO with the symbol

$$e^{ita_1^\lambda(x, \xi)} (D_t b^{(\lambda)} - \mathcal{L}_\lambda b^{(\lambda)}) \sim e^{ita_1^\lambda(x, \xi)} D_t \tilde{b}_0^{(\lambda)} + \sum_{k=1}^{\infty} e^{ita_1^\lambda(x, \xi)} (D_t \tilde{b}_k^{(\lambda)} - \mathcal{L}_\lambda \tilde{b}_{k-1}^{(\lambda)}).$$

Notice that (outside a neighbourhood of $|\xi| = 0$) $D_t \tilde{b}_0^{(\lambda)} = 0$ and, for all $k \in \mathbb{N}$,

$$D_t \tilde{b}_k^{(\lambda)}(t; x, \xi) = D_t \left[i \int_0^t \mathcal{L}_\lambda \tilde{b}_{k-1}^{(\lambda)}(s; x, \xi) ds \right] = \mathcal{L}_\lambda \tilde{b}_{k-1}^{(\lambda)}(t; x, \xi).$$

Therefore all the terms in this asymptotic series are equal to zero (outside a neighbourhood of $|\xi| = 0$) and we conclude that $(D_t - A^\lambda)V_\lambda \in L^{-\infty}(\mathcal{M}, \Omega^{1/2})$.

Substituting (6.22) in (6.20) and taking into account (6.21), we obtain the asymptotic expansion (6.15).

Finally, we want to show that $U_\lambda(t) = V_\lambda(t)$ modulo an operator with smooth kernel smoothly dependent on t . The following argument is a repetition of a technique outlined in [11, Section 20]. First of all, set

$$R(t) = U_\lambda(-t)V_\lambda(t) - I.$$

Differentiate it with respect to t to obtain

$$\begin{aligned} D_t R_\lambda(t) &= -(D_t U_\lambda)(-t)V(t) + U_\lambda(-t)(D_t V_\lambda)(t) \\ &= -A^\lambda U_\lambda(-t)V_\lambda(t) + U_\lambda(-t)A^\lambda V_\lambda(t) + U_\lambda(-t)K(t). \end{aligned} \tag{6.23}$$

Since $U_\lambda(t)A^\lambda = A^\lambda U_\lambda(t)$, it follows from (6.23) that

$$D_t R_\lambda(t) = U_\lambda(-t)K(t) \tag{6.24}$$

and consequently $D_t R(t)$ is an operator with smooth kernel uniformly in t . Now,

$$\begin{aligned} \tilde{R}_\lambda(t) &= V_\lambda(t) - U_\lambda(t) \\ &= U_\lambda(t)R_\lambda(t). \end{aligned}$$

But $\tilde{R}_\lambda(0) = V_\lambda(0) - U_\lambda(0)$ and $D_t R_\lambda(t)$ is an operator with smooth kernel uniformly in t . Therefore we conclude that $V_\lambda(t) - U_\lambda(t)$ is an operator with smooth kernel uniformly in t . \square

We conclude by presenting the central result of this section. Combining the Fourier inversion formula for the operator $\omega(A)$ and Proposition 6.11, we obtain the desired

result for $\omega(A)$, with $\omega \in S_\rho^m(\mathbb{R})$ and $\rho > 1/3$. Precisely, it is a Ψ DO from the class $\omega(A) \in L_{\rho, \frac{1-\rho}{2}}^m(\varphi, \mathcal{M}, \Omega^{1/2})$ and its principal φ -symbol is $\omega(a_1)$ under the assumption that $\nabla_{\varphi, x} a_1 = 0$.

Theorem 6.15. *Let $\omega \in S_\rho^m(\mathbb{R})$ and $\rho \in (1/3, 1]$. If $\nabla_{\varphi, x} a_1 = 0$, then $\omega(A) \in L_{\rho, \frac{1-\rho}{2}}^m(\varphi, \mathcal{M}, \Omega^{1/2})$ and the symbol of $\omega(A)$ admits the asymptotic expansion*

$$\sigma_{\omega(A)}(x, \xi) \sim \omega(a_1(x, \xi))\sigma_{\varphi, I}(x, \xi) + \sum_{j=1}^{\infty} c_j(x, \xi)\omega^{(j)}(a_1(x, \xi)), \quad (6.25)$$

where $c_j \in \mathcal{S}^{j/3}(T^*\mathcal{M})$ and $\omega^{(j)}(\theta) = \partial_\theta^j \omega(\theta)$.

Remark 6.16. We proved in Proposition 5.10 that $\sigma_{\varphi, I}(x, \xi) = 1$ modulo $\mathcal{S}^{-1}(T^*\mathcal{M})$, therefore we have

$$\sigma_{\omega(A)}(x, \xi) \sim \omega(a_1(x, \xi)) \quad \text{mod } \mathcal{S}_{\rho, \frac{1-\rho}{2}}^{m-\rho+1/3}(\varphi, T^*\mathcal{M}).$$

The proof of Theorem 6.15 is a modification of the proof of [16, Theorem 11.2].

Proof. Because of conditions (6.11) and (6.12), we can assume without loss of generality that $\text{supp } \omega \subseteq (0, +\infty)$. Let us fix $\lambda \in ((1-\rho), 2/3)$ and set $\omega_\lambda(\theta) = \omega(\theta^{1/\lambda})$. Then $\omega_\lambda \in S_{1-(1-\rho)/\lambda}^{m/\lambda}(\mathbb{R})$, where $1 - (1-\rho)/\lambda > 0$.

Since $\partial_\theta^k \omega(\theta) \in S_\rho^{m-k\rho}$, then for k large enough $t^k \hat{\omega}(t) \in \mathcal{C}^{N_k}$ outside a neighbourhood of $t = 0$, where $\lim_{k \rightarrow \infty} N_k = \infty$. Therefore, $\hat{\omega}_\lambda(t)$ coincides with a rapidly decreasing function for large t , and, by the Fourier inversion formula, we have

$$\omega(A) = \frac{1}{2\pi} \int \hat{\omega}_\lambda(t) e^{itA^\lambda} dt,$$

where the integral converges in the weak operator topology. Let $\chi \in \mathcal{C}_0^\infty(\mathbb{R})$, such that $\chi(t) = 1$ in a neighbourhood of $t = 0$ and $\chi(t) = 0$ for large t . Then by

integration by parts,

$$\int (1 - \chi(t)) \hat{\omega}_\lambda(t) e^{itA^\lambda} dt = A^{-k\lambda} \int (i\partial_t)^k [(1 - \chi(t)) \hat{\omega}_\lambda(t)] e^{itA^\lambda} dt$$

for all positive integers k . Since $A^{-k\lambda} \in L^{-k\lambda}(\mathcal{M}, \Omega^{1/2})$, then $\int (1 - \chi(t)) \hat{\omega}_\lambda(t) e^{itA^\lambda} dt$ is a smoothing operator. Therefore, we have

$$\omega(A) = \int \chi(t) \hat{\omega}_\lambda(t) e^{itA^\lambda} dt \quad \text{mod } L^{-\infty}(\mathcal{M}, \Omega^{1/2}).$$

By Proposition 6.11, it follows that $\omega(A)$ is a Ψ DO with symbol

$$\sigma_{\omega(A)}(x, \xi) = \int e^{ita_1^\lambda(x, \xi)} \hat{\omega}_\lambda(t) \chi(t) b^{(\lambda)}(t; x, \xi) dt. \quad (6.26)$$

Substituting the asymptotic expansion (6.15) into (6.26), we get

$$\sigma_{\omega(A)}(x, \xi) \sim \omega(a_1(x, \xi)) \sigma_{\varphi, I}(x, \xi) + \sum_{j=1}^{\infty} \omega_{\lambda, j}(a_1(x, \xi)) b_j^{(\lambda)}(x, \xi),$$

where

$$\omega_{\lambda, j}(s) = d^j(\omega(r^{1/\lambda})) / dr^j \Big|_{r=s^\lambda}.$$

This implies (6.25) with some symbols $c_j \in \mathcal{S}^{j/3}(T^*\mathcal{M})$. □

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